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AN

ELEMENTARY TREATISE

OF

SPHERICAL GEOMETRY

AND

TRIGONOMETRY.

BY

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SPHERICAL GEOMETRY.

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SPHERICAL GEOMETRY.

DEFINITIONS.

1. A *sphere* is a solid such that all points in its surface are equidistant from a certain point within called the *center*.

2. A *radius* of a sphere is any straight line drawn from the center to the surface.

All radii of a sphere are equal.

3. A sphere may be described by the revolution of a semicircle about its diameter, the middle of the diameter being the center, and half the diameter a radius of the sphere.

4. A *diameter* of a sphere is any straight line passing through the center and terminating each way in the surface.

All diameters of a sphere are equal, each of them consisting of two radii.

5. The *axis* of a sphere is a diameter about which the sphere is supposed to have been described by the revolution of a semicircle.

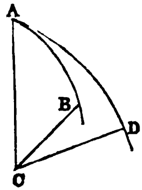
6. Every intersection of a plane with a sphere is a *circle*, as will be seen from the demonstration of Prop. VI.

7. The intersection of a sphere with a plane passing through the center is called a *great* circle, and its intersection with any other plane, a *small* circle.

8. The *axis* of a *circle* of a sphere, is that diameter of the sphere which is perpendicular to the circle.

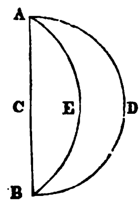
The extremities of the axis are called the *poles* of the circle.

9. The angle made by the arcs of two great circles is called a *spherical angle*, and is to be regarded as the same with the angle between the *planes* of the circles.



Thus, BAD is a spherical angle, having for its substitute the angle between the planes ACB and ACD, supposing C to be the center of the sphere.

10. A *spherical lune* is a part of the surface of a sphere included between two great semicircles having a common diameter, as ADBE.



A *spherical ungula* or *wedge* is a part of a sphere, bounded by a lune and the two great semicircles which include the lune, as CADBE.

11. A *spherical triangle* is a part of the surface of a sphere, included between the arcs of three great circles.

The arcs are called *sides* of the triangle.

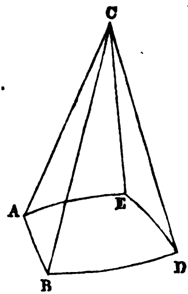
12. Spherical triangles are distinguished as *right-angled*, *isosceles*, *equilateral*, &c., in the same way as plane triangles.

A *quadrantal* triangle is that of which one side is a quadrant.

13. A *spherical polygon* is a portion of the surface of a sphere, bounded by several arcs of great circles; which arcs are called *sides* of the polygon.

14. Each side of a triangle or a polygon must be understood to be *less* than a *semicircumference* of a great circle, unless the contrary is stated.

15. A *spherical pyramid* is a part of the sphere, contained by the planes of a solid angle whose vertex is the center, and the spherical polygon included by these planes; as ABCDE.



The polygon is called the *base* of the pyramid.

When the base is a spherical triangle, the pyramid is called *triangular*.

16. A line or plane is said to *touch* or *be tangent to* a sphere, when it meets the surface of the sphere in one point only.

And two spheres are said to *touch* each other, when they meet and do not intersect.

STRAIGHT LINE AND SPHERE.

Prop. I.

If a perpendicular drawn from the center of a sphere to any straight line be equal to the radius of the sphere, this line touches the sphere at the foot of the perpendicular.

For since the perpendicular is equal to the radius, the foot of the perpendicular is in the surface of the sphere; the line therefore *meets the surface* at the foot of the perpendicular: and every other point in the line is *without the surface*, being further from the center of the sphere, because the perpendicular is the shortest distance from a point to a straight line.

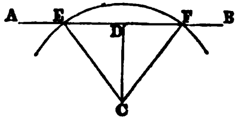
Prop. II.

If a perpendicular from the center of a sphere to any straight line be longer than the radius, the line is wholly without the sphere.

For no point of the line can be at less than the perpendicular distance from the center: but this is greater than the radius; wherefore the line must at every point be without the surface of the sphere.

Prop. III.

If a perpendicular from the center of a sphere to any straight line be less than the radius, the line will meet the surface in two points on opposite sides of the perpendicular, and at equal distances from it; the part of the line between these points being within the surface of the sphere, and the rest of the line without it.



Let AB be the straight line, C the center of the sphere, and CD the perpendicular, which is less than the radius. In the plane passing through AB and C, describe an arc of a circle from the center C, with a radius equal to that of the sphere, and let it meet AB in E and F.

Then as the distances CE and CF are each equal to the radius of the sphere, the points E and F are in the surface; and (Euc. 3, 3) ED is equal to DF.

Also any point in AB between E and F is within the surface of the sphere, since its distance from C is less than CE or CF; and any point in EF produced either way, is further from C than F or E, and therefore is without the surface of the sphere.*

PLANE AND SPHERE.

Prop. IV.

If the perpendicular from the center of a sphere to any plane be equal to the radius of the sphere, the plane is a tangent to the sphere at the foot of the perpendicular.

For as the perpendicular is equal to the radius, the foot of it is a point in the surface of the sphere, and the plane *meets the surface* in this point. Moreover it can meet it in no other point: for the distance from any other point in the plane to the center of the sphere is greater than the perpendicular, and therefore greater than the radius of the sphere; so that every such point is without the surface.

Prop. V.

If a perpendicular from the center of a sphere to a plane be longer than the radius of the sphere, the plane is wholly without the sphere.

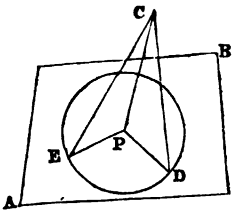
For this perpendicular is the shortest distance from the center to the plane; and as the distance from the

* The limits of this treatise are such as to require the omission here of a number of obvious propositions concerning the straight line and sphere, similar to several that are given for the circle in the third book of Euclid; as for example, the propositions 7, 8, 14, 15, 35, 36, and 37 of that book.

center to every point of the plane is therefore greater than the radius, the plane must at every point be without the spherical surface.

Prop. VI.

If a perpendicular to a plane from the center of a sphere be shorter than the radius, the plane will cut the sphere, the section being a circle whose center is the foot of the perpendicular.



Let C be the center of the sphere, AB the plane, and CP the perpendicular to it. As CP is by supposition less than the radius, P must be within the surface of the sphere: from P draw any line PD at right angles to CP, and meeting the surface in D. Then a circle described in the plane AB, from P as a center, and with a radius equal to PD, will be the intersection of that plane with the sphere.

Draw in this circle any radius PE, and join CE, CD: the angle CPE is a right angle (Euc., Suppl. 2, Def. 1); and in the right-angled triangles CPE and CPD, as the sides PE and PD are equal, and CP is a common side, CE and CD are equal: but as D is a point in the surface of the sphere, CD is a radius; wherefore CE is also a radius, and the point E is in the surface. In like manner may it be shown that any other point in the circumference of the circle described about P as a center in the plane AB, as in the surface of the sphere: this circumference therefore is the intersection of the plane with the *surface* of the sphere, and the circle itself is the intersection of the plane and sphere.

Cor. The nearer the cutting plane is to the center of the sphere, the larger will be the circular intersection. For the square of the radius of the circle added to the square of the perpendicular from the center of the sphere, is equal to the square of the radius of the sphere; thus, $EP^2+PC^2=CE^2$: whence the radius of the circle must be greater, as the perpendicular from the center is less.

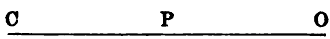
The circle is greatest when its plane passes through the center of the sphere; in which case, as has been stated (Def. 7), it is called a great circle.

Every two great circles of a sphere bisect each other; for the center of the sphere is the center of each, and their common section is a diameter of each.

TWO SPHERES.

Prop. VII.

If the distance between the centers of two spheres be equal to the sum of their radii, the spheres will touch each other externally, the point of contact being in the line which joins their centers.



Let C, O, be the centers of the spheres, and let the distance CO equal the sum of their radii, CP being equal to the radius of the sphere whose center is C, and OP consequently equal to the other radius. Then the point P must be in the surface of each sphere. And this is the only point common to the spheres: for the lines drawn from any point to C and O, must at least be equal to PC and PO together, since CPO is the shortest distance between C and O: and if the point be *within* the surface of the sphere whose center is C, then as its distance from C is less than CP, its distance from O must be greater than OP, so that the point must be *without* the sphere whose center is O: again, if the point be *in* the surface of the sphere whose center is C, its distance from C equals CP, and its distance from O therefore either exceeds OP, so that the point is without the sphere whose center is O, or is equal to OP: but P is evidently the only point whose distance from C equals PC, and whose distance from O equals PO. The spheres therefore touch each other externally at P.

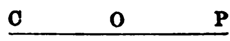
PROP. VIII.

If the distance between the centers of two spheres be greater than the sum of their radii, the spheres will be wholly exterior to each other.

For as the straight line that joins the centers is the shortest line between them, the distances from any point to these centers must together be greater than the sum of the radii of the spheres. And if any point be taken in one of the spheres, then as its distance from the center of this sphere is not greater than the radius, its distance from the center of the other sphere must be greater than the radius of that sphere, and the point must accordingly be without the surface.

Prop. IX.

If the distance between the centers of two spheres be equal to the difference of their radii, the smaller sphere will touch the other internally, the point of contact being in the straight line which passes through their centers.



Let C, O, be the centers of two spheres, OP being a radius of the smaller sphere, and COP a radius of the larger: the spheres will touch each other internally at the point P.

For as P is the extremity of a radius of each sphere, it is in the surface of each, and thus is common to the two spheres. And P is the only point of the smaller sphere that is not within the surface of the larger: for the distance of any point from the center C of the larger sphere, cannot exceed the distance of the same point from O, added to the line CO (Euc. 1, 20): and if the point be *within the surface* of the smaller sphere, then as its distance from O is less than OP, this distance added to CO is less than CP, wherefore the distance of the point from C is less than CP, which is a radius of the larger sphere, and the point is accordingly within the surface of this sphere. Again, if the point be *in the surface* of the smaller sphere, its distance from O equals OP, and this distance added to CO equals CP: the distance of the point from C is therefore less than CP the radius of the larger sphere, so that the point is within the surface of this sphere; or if the distance is not less, it is equal to CP: but evidently there is no other point than P, whose distances from C and O are equal to PC and PO. Since then the spheres meet at P, and this is the only point of the smaller sphere which is not *within* the surface of larger, they touch each other internally at this point.

Prop. X.

If the distance between the centers of two spheres is less than the difference of their radii, the smaller sphere is wholly within the surface of the other.

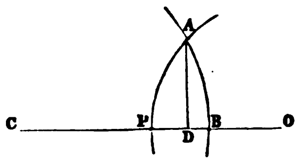
Let C be the center of the larger and O the center of the smaller of two spheres; CO the distance between the centers being less than the difference of the radii of the spheres: the smaller sphere is wholly within the other.

This is to be proved by showing that every point of the smaller sphere is within the surface of the larger, or what amounts to this, that the distance from any point whatever of the smaller sphere to the center of the larger, is less than the radius of the latter.

Now as the distance from any point of the smaller sphere to its center O, is not greater than the smaller radius, and the distance CO is less than the difference of the radii of the spheres, the sum of those two distances is less than the larger radius; but this sum is not less than the distance of the point from C the center of the larger sphere (Euc. 1, 20); wherefore this distance is less than the radius of the larger sphere; as was to be proved.

Prop. XI.

If the distance between the centers of two spheres be less than the sum but greater than the difference of their radii, the surfaces of the spheres will intersect in the circumference of a circle perpendicular to the line which joins the centers, and having its center in that line.



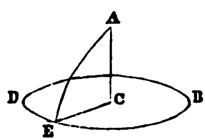
Let C, O be the centers of two spheres; CB, OP their radii, whose sum is greater than CO, and whose difference is less; the surfaces of these spheres will cut each other in the circumference of a circle at right angles to CO, and having its center in this line.

With the radii CB, OP, draw in any plane passing through CO, arcs of circles meeting in A: draw AD perpendicular to CO, and let a plane pass through D at right angles to CO; then with D as a center and with a radius equal to DA describe a circle in that plane: this will be the intersection of the plane with either sphere, as may be seen from the demonstration of Prop. VI: its circumference is therefore the common section of the surfaces of the spheres; and the circle is perpendicular to CO and has its center in this line; as was to be proved.

POLES OF SPHERICAL CIRCLES.

Prop. XII.

The arc of a great circle between the pole and the circumference of another great circle is a quadrant; and is at right angles to that circumference.



Let DEB be a great circle, A its pole, and AE an arc of a great circle drawn from A to the circumference of the circle DEB; the arc AE is a quadrant.

From C the center of the sphere, draw CA and CE. Because A is the pole of the circle DEB, AC is perpendicular to the plane of that circle, and therefore perpendicular to CE which it meets in this plane; and as the angle ACE is a right angle, the arc AE which subtends it, is a quadrant.

Again, the arc AE is at right angles to the arc DEB: for the spherical angles AED, AEB are the angles made by the plane ACE with the plane of the circle DEB: and these planes are at right angles to each other; because one of them, namely ACE, passes through AC a line perpendicular to the other.

Cor. The distances from all points in the circumference of a great circle to its pole are equal, as each is the chord of a quadrant of a great circle.

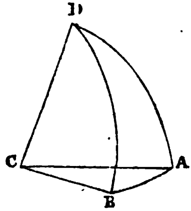
Prop. XIII.

If two great circles intersect at right angles, each passes through the poles of the other.

For if a diameter be drawn in either circle at right angles to the common section of the two, it will be perpendicular to the other circle, and its extremities will therefore be the poles of this other circle.

Prop. XIV.

If the arcs of two great circles drawn from a point in the surface of a sphere to the circumference of another great circle be quadrants, the point is a pole of this great circle.



Let AD and BD, arcs of great circles drawn from D to the circumference AB of another great circle, be quadrants, then will D be the pole of the great circle of which AB is an arc.

Since AD and DB are quadrants, ACD and BCD are right angles; and since DC is at right angles to the lines CA and CB, it is at right angles to the plane ACB; which is the plane of the great circle to which the arc AB belongs; wherefore the point D is a pole of that circle (Def. 8); as was to be proved.

Prop. XV.

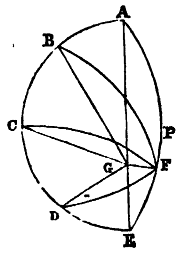
The circumferences of two great circles at right angles to a third great circle, will meet in the pole of this circle.

For the circumference of each of the first two circles will pass through the poles of the third (Prop. 13), and these poles must accordingly be the points in which the two circumferences intersect.

Prop. XVI.

Of arcs of great circles drawn from the circumference of another great circle to a point in the surface of a sphere which is not the pole of that circle, the greatest is that which passes through the pole, and one that is nearer than another to this, is greater than that other.

Let P be the pole of a great circle ACE, and F another point in the surface of the sphere; of arcs drawn from F to the circumference ACE, FA which passes through P is the greatest, and FB which is nearer than FC to FA, is greater than FC.



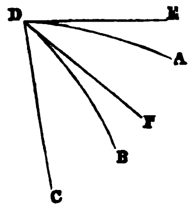
Let AE be the common section of the great circles ACE, APE; draw FG perpendicular to AE, and join GB, GC, GD.

Because P is the pole of ACE, the great circle APE is at right angles to ACE, and FG which is perpendicular to the common section AE, is perpendicular to the plane ACE, and to the lines GA, GB, GC, GD, which it meets in that plane. Now since the center of the circle ACE is in AG, AG is the longest line that can be drawn from G to the circumference; and GB is greater than GC, GC than GD, &c. And since AGF is a right angle, the sum of the squares of AG and GF is equal to the square of the chord (not shown in the fig.) of the arc AF: likewise the squares of BG and GF are together equal to the square of the chord BF. But as AG is greater than BG, AG^2+GF^2 is greater than BG^2+GF^2; wherefore the chord AF is greater than the chord BF, and the arc AF than the arc BF. In like manner may it be shown that the arc BF is greater than CF, CF than DF, &c.

SPHERICAL ANGLES.

PROP. XVII.

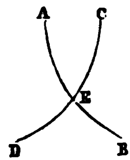
The spherical angle made by two arcs of great circles, is the same as the angle made by the tangents of those arcs at their point of intersection.



Let AD, BD be arcs of great circles intersecting in D, and let DE and DF be tangents to those arcs; the spherical angle ADB is the same as the angle EDF.

For the angle ADB is (Def. 9) the same as the angle between the planes of the arcs AD and BD, which planes are ADC and BDC, C being the center of the sphere; and EDF is the angle between those planes, since DC is their common section, and DE, DF are perpendicular to DC, the one in the plane ADC, and the other in the plane BDC (Euc. 3, 18).

Cor. If two arcs of great circles cut each other, their *vertical* or *opposite* angles are equal. Thus, the arcs AB and CD intersecting in E, make the opposite angles AEC, DEB, equal to each other: for either of these angles is the same as the angle made by the tangents to the arcs at E.

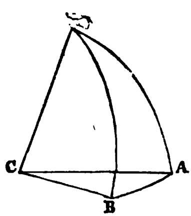


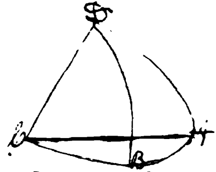
Also the sum of two *adjacent* spherical angles, as AEC and BEC, is two right angles.

Prop. XVIII.

The spherical angle made by two arcs of great circles, is measured by the arc which they intercept on the circumference of a great circle whose pole is the angular point.

Let ADB be a spherical angle made by AD and BD, arcs of great circles in a sphere whose center is C, and let AB be an arc of the great circle whose pole is D; AB is the measure of the spherical angle at D. For since D is the pole of the great circle of which AB is an arc, the radius DC is perpendicular to the plane of that circle, and therefore at right angles to the



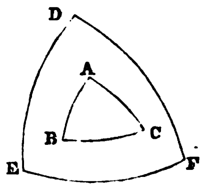


lines CA and CB which it meets in that plane; so that (Euc. Suppl. 2, Def. 4) BCA is the angle between the planes ACD and BCD, which angle is the same as the spherical angle ADB: and as AB is the measure of the angle ACB, it is also the measure of the angle ADB; as was to be proved.

SPHERICAL TRIANGLES.

Prop. XIX.

If about the vertices of a triangle, as poles, arcs of great circles be drawn, they will form a triangle whose verticles are poles of the great circles passing through the sides of the first triangle.



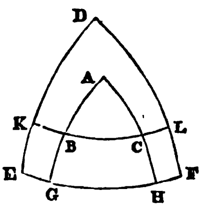
Let ABC be a spherical triangle, and about A, B, C, as poles, let arcs of great circles EF, FD, DE, be drawn, forming the triangle DEF; the vertices D, E, F, are respectively poles of the great circles passing through BC, CA, AB.

For since B is the pole of the great circle DF, the arc of a great circle drawn from B to D is a quadrant (Prop. 12); and since C is the pole of DE, the arc drawn from C to D is also a quadrant. Now when the arcs of two great circles drawn from a point in the surface of a sphere to the circumference of another great circle are quadrants, the point is the pole of this circle; therefore D is the pole of the great circle passing through BC. In like manner may it be shown that E is the pole of the circle AC, and F the pole of AB.

Def. Such triangles as those above mentioned are called *polar* triangles, each being said to be polar to the other.

Prop. XX.

The sides of a spherical triangle are supplements to the measures of the angles of its polar triangle.



Let ABC, DEF, be polar triangles, A, B, C, being poles of EF, FD, DE, respectively, and D, E, F, poles of BC, CA, AB; the sides of the triangle ABC are supplements to the measures of the angles D, E, F, and the sides of the triangle DEF are likewise supplements to the measures of the angles A, B, C, of the triangle ABC.

Let the arcs AB, AC, produced if necessary, meet EF in G and H; and let the arc BC meet DE and DF, in K and L.

Then, as A is the pole of the great circle EGHF, AG and AH are quadrants (Prop. 12); and GH is the measure of the spherical angle at A (Prop. 18); also GF and HE are quadrants, because E and F are poles of AC and AB respectively; wherefore the arcs GF and HE together are equal to the semi-circumference of a great circle: but these two arcs are together equal to EF and GH; hence EF is the supplement of GH, that is, the supplement to the measure of the angle BAC. In like manner it may be shown that FD, DE are supplements to the measures of the angles ABC, ACB, respectively.

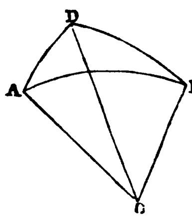
Again, because D is the pole of the great circle passing through BC, KL is the measure of the spherical angle EDF; and since B and C are the poles of DF and DE respectively, BL and CK are quadrants, and are together equal to the semi-circumference of a great circle: but their sum is the same as that of BC and KL; wherefore BC is the supplement of KL, the arc which measures the spherical angle at D. In like manner may it be shown that the other sides AB, AC, of the triangle ABC, are respectively supplements to the measures of the angles F and E of the triangle DEF.

Schol. From this property of polar triangles, they are often called *supplemental* triangles.

Prop. XXI.

Any two sides of a spherical triangle are together greater than the third side.

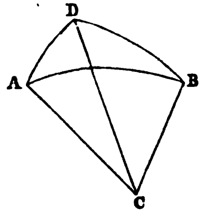
Let ABD be a spherical triangle; any two of its sides, as AD and DB, are together greater than the third side AB.



Let C be the center of the sphere: the planes of the great circles passing through the three sides of the triangle form a solid angle at C; and of the plane angles containing this solid angle, any two, as ACD and BCD, are together greater than the third ACB (Euc. Suppl. 2, 20); wherefore the arcs AD and BD which measure the first two angles, are together greater than AB which measures the third.

Prop. XXII.

The three sides of a spherical triangle are together less than the circumference of a great circle.



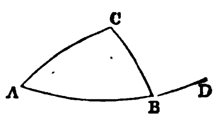
Let ABD be a spherical triangle; the sum of the sides AB, BD, and DA, is less than the circumference of a great circle.

For if C be the center of the sphere, the arcs AB, BD, DA, are respectively the measures of the plane angles ACB, BCD, DCA; which are angles containing a solid angle at C, and therefore are together less than four right angles (Euc. Suppl. 2, 21). Hence the arcs which measure these angles, namely, AB, BD, DA, are together less than four quadrants or the circumference of a great circle.

Cor. In like manner it may be proved that the sides of any spherical *polygon* are together less than the circumference of a great circle.

Prop. XXIII.

Each of the angles of a spherical triangle is less than two right angles, and the sum of the three angles is greater than two right angles.



Let ABC be a spherical triangle; each of its angles A, B, and C, is less than two right angles, and the sum of the three is greater than two right angles.

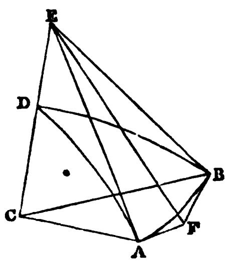
Produce the side AB, so as to make an angle CBD adjacent to ABC; these angles are together equal to two right angles (Cor. Prop. 17); therefore ABC, one of them, is less than two right angles: in like manner it may be shown that A and C, the other two angles of the triangle, are each less than two right angles.

Again, the sides of the triangle that is polar to ABC, are supplements to the measures of the angles A, B, C; wherefore those sides together with the measures of these angles are equal to three semi-circumferences of great circles; but the sides of the polar triangle are together less than the circumference, or two semi-circumferences of a great circle (Prop. 22); wherefore the measures of the angles A, B, C, are together greater than one semi-circumference or two quadrants, and the angles accordingly greater than two right angles.

Prop. XXIV.

If two sides of a spherical triangle are equal, the angles opposite them are also equal.

Let ABD be a spherical triangle of which the sides AD and BD are equal; then will the angle DBA be equal to the angle DAB.

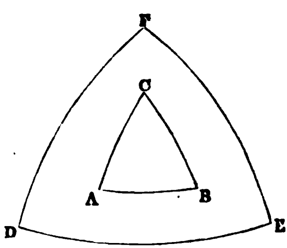


If AD and BD are quadrants, the angles DBA and DAB are equal, each of them being a right angle (Props. 14, 12); but if not, from C the center of the sphere, draw the radii CA, CB, CD; in the plane CAB, draw AF and BF, tangents to the arc AB; and in the plane CAD draw AE a tangent to AD, meeting CD produced in E: also join EB, EF.

Then, as the arcs AD and BD are equal, the angles ACD and BCD subtended by them, are also equal; and the triangles ACE, BCE, having two sides and the included angle of one equal to two sides and the included angle of the other, are every way equal, the side BE being equal to AE, and the angle EBC to the angle EAC; but EAC is a right angle (Euc. 3, 18); therefore EBC is a right angle, and EB is a tangent to the arc BD. Again, in the triangles EAF and EBF, FA and FB are equal (Euc. 3, 36), AE and BE are equal, as has been proved above, and EF is a common side; wherefore the angle EAF is equal to the angle EBF: but these angles are the same as the spherical angles DAB, DBA (Prop. 17); which are therefore also equal.

Prop. XXV.

If two angles of a spherical triangle are equal, the sides opposite them are also equal.



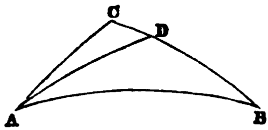
Let ABC be a spherical triangle, of which the angles A and B are equal; the sides AC and BC are also equal.

Draw the triangle FDE which is polar to ABC, FE and FD being arcs of great circles to which A and B respectively are poles. Then as the arcs FE and FD are supplements to the measures of the equal angles A and B (Prop. 20), they are equal; wherefore the angles

FDE and FED are also equal (Prop. 24): again, the sides AC and BC are supplements to the measures of the angles E and D (Prop. 20); and as these angles have been shown to be equal, the arcs AC and BC are also equal: which was to be proved.

Prop. XXVI.

If two angles of a spherical triangle are unequal, the greater has the greater side opposite it.



Let ABC be a spherical triangle, in which the angle CAB is greater than CBA; the side BC, opposite the greater angle, is longer than the side AC.

Through A the vertex of the greater angle, draw AD an arc of a great circle cutting off the angle BAD equal to the angle at B: then as the angles DAB and DBA of the triangle ABD are equal, the opposite sides AD and DB are equal (Prop. 25); hence BC, which is the sum of BD and DC, equals the sum of AD and DC; but AD and DC are together greater than AC (Prop. 21); therefore BC is greater than AC.

Prop. XXVII.

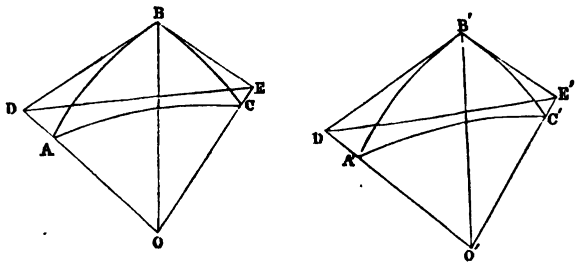
If two sides of a spherical triangle are unequal, the greater has the greater angle opposite it.

For if the angle opposite the longer side is not greater than the angle opposite the other; the former must either be equal to the latter or be less than it; but the angles cannot be equal, for equal angles are subtended by equal sides; whereas the sides are unequal: nor can the angle opposite the longer side be less than the other angle; for if two angles of a triangle are unequal, the greater has the longest side opposite it (Prop. 26): the angle opposite the longer of the unequal sides of a spherical triangle is therefore greater than the angle opposite the other.

Prop. XXVIII.

If on the same sphere or on equal spheres, the sides of one triangle are severally equal to those of another, the angles of the one are likewise equal to the angles of the other.

Let ABC be a triangle in the surface of a sphere whose center is O, and A′B′C′ another triangle in the surface of the same or an equal sphere whose center is O′; also let AB be equal to A′B′, BC to B′C′, and AC to A′C′; then will the angle A be equal to A′, B to B′, and C to C′.



Draw BD, BE, tangents to the arcs BA, BC, in the planes AOB, BOC, and meeting the radii OA and OC produced, in D and E; join also BO and DE: and in like manner construct a figure about the triangle A′B′C′.

Then in the triangles BDO, B′D′O′, the angles O and O′ are equal, being subtended by the equal arcs AB and A′B′; and the angles DBO and D′B′O′ are equal, each being a right angle (Euc. 3, 18); also the sides OB and O′B′ are equal: wherefore the remaining sides of one triangle are equal to the remaining sides of the other; BD being equal to B′D′, and DO to D′O′. In like manner it may be proved that BE is equal to B′E′, and EO to E′O′. Now in the triangles DOE, D′O′E′, the two sides DO, OE, being equal to D′O′ and O′E′, respectively, and the angles DOE, D′O′E′ being equal, because subtended by equal arcs AC, A′C′, the third side DE must be equal to the side D′E′. Then in the triangles BDE, B′D′E′, the sides of the former are severally equal to those of the latter; wherefore the angle DBE opposite the side DE, is equal to the angle D′B′E′ opposite the side D′E′. But DBE is equal to the spherical angle ABC, and D′B′E′ to A′B′C′ (Prop. 17): therefore the spherical angles ABC and A′B′C′ are equal. In a similar way it may be shown that the angle A is equal to A′ and C to C′.

Prop. XXIX.

If on the same sphere or on equal spheres, the angles of one triangle be severally equal to those of another, the sides of the one will likewise be equal to the sides of the other.

Let the triangles be called M and M′; and let the triangle which is polar to M be denoted by N, and that which is polar to M′, by N′. Then, since the sides of the triangles N and N′ are supplements to the measures of the angles of M and M′ (Prop. 20), and since the angles of M are severally equal to those of M′, the sides of the triangle N must be severally equal to those of N′, and hence, by the last proposition, the angles of N must likewise be equal to the angles of N′. But the sides of the triangles M and M′ are supplements to the measures of these angles (Prop. 20); wherefore the sides of the triangle M must severally be equal to those of M′.

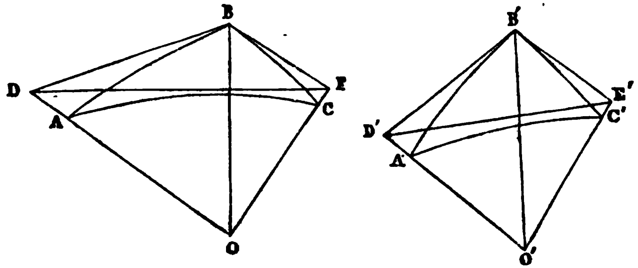
Prop. XXX.

If, on the same sphere or on equal spheres, two sides of one triangle be equal to two sides of another, each to each, and the base of the one be greater than the base of the other, the angle opposite the base in the former triangle will be greater than the corresponding angle in the latter.

Let AB, BC, two sides of a triangle ABC on a sphere whose center is O, be respectively equal to A′B′, B′C′, two sides of another triangle A′B′C′ on the same or an equal sphere whose center is O′; and let the base AC of the former triangle be greater than A′C′ the base of the latter; then will the angle ABC opposite AC, be greater than the angle A′B′C′ opposite A′C′.

Draw BD, BE, tangents to the arcs BA, BC, and meeting the radii OA and OC produced, in D and E; join also BO, DE: and in a similar way construct a figure about the triangle A′B′C′.

Then in the triangles DBO, D′B′O′, as the angles at O and O′ opposite equal arcs AB and A′B′ are equal, and the angles DBO, D′B′O′, are right angles, and the sides BO, B′O′ are equal, the remaining sides of one triangle must be equal respectively to those of the other, namely, BD to B′D′ and DO to D′O′. In like manner it may be shown that BE is equal to B′E′ and EO to E′O′.

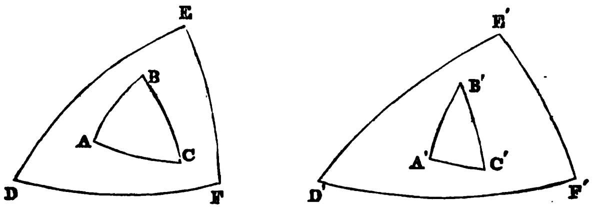


Now in the triangles DOE, D′O′E′, since the two sides DO, OE, are equal to D′O′, O′E′, each to each, and the angle DOE is greater than D′O′E′, being subtended by a greater arc, therefore the base DE is greater than D′E′. Again, in the triangles BDE, B′D′E′, since the sides DB, BE, are respectively equal to D′B′, B′E′, and the base DE is greater than D′E′, therefore the angle opposite DE, namely DBE, is greater than the angle D′B′E′ opposite D′E′. But those angles are the same as the spherical angles ABC, A′B′C′; wherefore the angle ABC is greater than A′B′C′; which was to be proved.

Prop. XXXI.

If on the same sphere or on equal spheres, two angles of one triangle be equal to two angles of another, each to each, and the third angle of the one be greater than the third angle of the other, the side opposite the third angle of the former triangle is greater than the corresponding side of the latter.

Let ABC and A′B′C′ be two triangles on equal spheres having the angles A and C equal to A′ and C′, respectively, but the angle B greater than B′; then will the side AC opposite B be greater than A′C′ opposite B′.



Describe the triangle DEF polar to ABC, A, B and C being the poles of EF, DF and ED respectively; and in like manner describe the triangle D′E′F′ polar to A′B′C′. Then as EF and E′F′ are supplements to the measures of equal angles A and A′, they are also equal; for a like reason, ED and E′D′ are equal: but DF is less than D′F′, because it is the supplement to the measure of a greater angle, B being greater than B′. Now since DEF and D′E′F′ are two spherical triangles in which the side DE is equal to D′E′, EF equal to E′F′, and DF less than D′F′, the angle DEF opposite DF must be less than D′E′F′ opposite D′F′ (Prop. 30): but AC is the supplement to the measure of the angle E (Prop. 20), and A′C′ the supplement to the measure of E′: therefore AC is greater than A′C′; which was to be proved.

Prop. XXXII.

If two triangles on the same sphere or on equal spheres have two sides and the included angle of one severally equal to two sides and the included angle of the other, *the remaining side and angles of the one will be equal to the corresponding side and angles of the other.*

The remaining sides cannot be unequal, for if they were so, the angles opposite them would be unequal (Prop. 30); whereas those angles are supposed to be equal. The sides are therefore equal; whence also the remaining angles of one triangle are equal to the corresponding angles of the other.

Prop. XXXIII.

If two triangles on the same sphere or on equal spheres have two angles and the intermediate side of one, severally equal to two angles and the intermediate side of the other, the remaining angle and sides of the one will be equal to the corresponding angle and sides of the other.

The remaining angles cannot be unequal, since the sides opposite them would then be unequal (Prop. 31); whereas those sides are supposed to be equal. The angles must therefore be equal; and the remaining sides of one triangle be accordingly equal to those of the other.

Prop. XXXIV.

If on the same sphere or on equal spheres, an angle of one triangle be greater than an angle of another, and the sides containing the former angle be equal to those that contain the latter, each to each, the third side of the former triangle will be greater than that of the latter.

Let the sides of one triangle be denoted by X, Y, Z, and let the angle contained by X and Y, or the angle opposite Z, be denoted by C, and let the corresponding parts of the other triangle be denoted by X′, Y′, Z′, C′, X and Y being equal respectively to X′ and Y′, and C being greater than C′; then will Z be greater than Z′.

Z and Z′ cannot be equal, for then would C be equal to C′ (Prop. 28): neither can Z be less than Z′; for then would C be less than C′ (Prop. 30). Z must therefore be greater than Z′; as was to be proved.

Prop. XXXV.

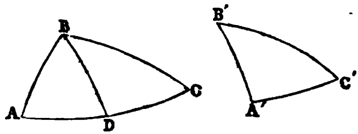
If on the same sphere or on equal spheres, a side of one triangle be greater than a side of another, and the angles adjacent to the former be equal to those adjacent to the latter, each to each, the third angle of the former triangle, will be greater than that of the latter.

Let the angles of one triangle be denoted by A, B and C, and the side to which A and B are adjacent, that is the side opposite C, by Z; let the corresponding parts of the other triangle be denoted by A′, B′, C′, Z′: let A, B, be equal to A′, B′, each to each; and let Z be greater than Z′; then will C be greater than C′.

The angles C and C′ cannot be equal; for then would Z be equal to Z′ (Prop. 29); neither can C be less than C′; for then would Z be less than Z′ (Prop. 31). The angle C must therefore be greater than C′; as was to be proved.

Prop. XXXVI.

If on the same sphere or on equal spheres, a side and the opposite angle in one triangle be severally equal to a side and the opposite angle in another, and a second side in the former triangle be equal to a second side in the *latter, the angles opposite these sides will also be equal, or else one will be the supplement of the other.*



Let ABC and A′B′C′ be two triangles on equal spheres, having the angles C and C′ equal, the opposite sides AB and A′B′ equal, and the sides BC and B′C′ also equal; then will the angles opposite these sides be equal, namely, the angles A and A′; or else the angle A will be the supplement of A′.

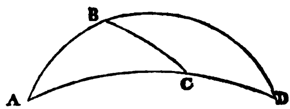
First, suppose the side AC to be equal to the side A′C′; then will the angles A and A′ be equal (Prop. 28 or Prop. 32).

Next let the sides AC and A′C′ be unequal; then will A be the supplement of A′. For supposing AC to be greater than A′C′, from AC cut off DC equal to A′C′, and draw the arc BD. Then will two sides and the included angle of the triangle BCD be severally equal to two sides and the included angle of B′C′A′; wherefore the third side BD will be equal to A′B′, and the angle BDC to the angle A′. But AB is equal to A′B′, therefore AB and BD are equal, and hence the angles A and ADB are equal. Now since the angles A and A′ are equal to ADB and BDC, each to each, and since these last angles are supplements of each other (Prop. 17, Cor.), the angles A and A′ are also supplements of each other; as was to be proved.

Cor. When the angles A and A′ are equal, but not right angles, the sides AC and A′C′ are equal, and also the angles ABC and B′. For when AC and A′C′ are unequal A is the supplement of A′; and if A is the supplement of A′ and at the same time equal to it, both are right angles. But they are not right angles; therefore AC and A′C′ are equal, and also the angles opposite them.

Prop. XXXVII. (*Lemma.*)

If two angles of a spherical triangle are supplements of each other, their opposite sides also are supplements of each other.



Let ABC be a spherical triangle, in which the angles B and C are supplements of each other; then will the sides AB and AC be supplements of each other.

Let AB and AC produced meet in D. Then DBC is the supplement of ABC, and is therefore equal to ACB. For a like reason also DCB is equal to ABC. And since the side BC is common to the two triangles ABC, DCB, the sides AB, AC, must be equal to CD, BD, respectively (Prop. 33). But CD is the supplement of AC (Cor. Prop. 6); therefore AB is also the supplement of AC; as was to be proved.

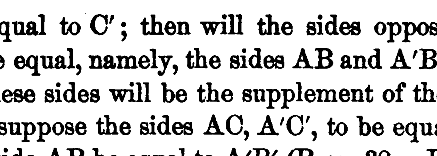
Cor. Conversely, if the sides of a spherical triangle are supplements of each other, the opposite angles also are supplements of each other. For if AB and AC are supplements of each other, DC must be equal to AB, each being the supplement of AC, and DB must be equal to AC, each being the supplement of AB: and then as the side BC is common to the two triangles

ABC, DBC, the angles ABC, ACB, are equal to BCD, CBD, each to each (Prop. 28), and as DBC is the supplement of ABC, ACB is also the supplement of ABC; which was to be proved.

Prop. XXXVIII.

If on equal spheres, a side and the opposite angle of one triangle be respectively equal to a side and the opposite angle of another, and a second angle of the former triangle be equal to a second angle of the latter, the sides opposite these angles will also be equal, or else one will be the supplement of the other.

Let ABC, A′B′C′, be two triangles in which the angle A and the opposite side BC are equal to the angle A′ and the opposite side B′C′, each to each, and the angle



C also equal to C′; then will the sides opposite these angles be equal, namely, the sides AB and A′B′: or else one of these sides will be the supplement of the other.

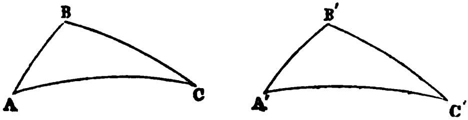
First, suppose the sides AC, A′C′, to be equal. Then will the side AB be equal to A′B′ (Prop. 32 or Prop. 33).

Next let the sides AC, A′C′, be unequal; then will AB be the supplement of A′B′. For supposing CA to be greater than C′A′, cut off CD equal to A′C′, and draw the arc BD. Then will there be in the triangles DBC, A′B′C′, two sides and the included angle of one equal to two sides and the included angle of the other; wherefore the third sides BD and A′B′ are equal, and the angle BDC is equal to A′. But A is equal to A′; therefore A is equal to BDC; and as BDC is the supplement of BDA, A is also the supplement of BDA. Since then in the triangle BDA, the angles at A and D are supplements of each other, their opposite sides BA and BD are supplements of each other. But BD is equal to B′A′, as was before proved; therefore BA and B′A′ are supplements of each other; as was to be shown.

Prop. XXXIX.

If in two right-angled spherical triangles, the hypotenuse and another side of one be equal to the hypotenuse and another side of the other, each to each, and if the hypotenuse be not equal to that other side in either triangle, the remaining side and angles of the one will be respectively equal to the remaining side and angles of the other.

Let ABC, A′B′C′, be two spherical triangles right-angled at B and B′, having the sides AB, AC, equal to A′B′, A′C′, each to each; and let the sides AB and AC be unequal; then will the side BC be equal to B′C′, and the angles A and C be equal to A′ and C′, respectively.



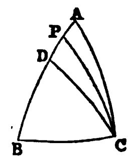
Since AB is equal to A′B′, and AC to A′C′, A′B′ is greater or less than A′C′, according as AB is greater or less than AC. If AB is less than AC, the angle C is less than B, that is, less than a right angle; and the angle C′ must for a like reason be less than a right angle; but if AB is greater than AC, the angles C and C′ will each be greater than a right angle. These angles then are each greater or each less than a right angle, and their sum is greater or less than two right angles. Now (Prop. 36) the angles C and C′ are equal, or they are supplements to each other; but as their sum is not equal to two right angles, they are not supplements of each other: they are therefore equal; and consequently the angle A is equal to A′, and the side BC to B′C′ (Prop. 36, Cor.).

Schol. The case in which AB is equal to AC, remains to be considered. In this case, the angle at C being equal to B, is a right angle, A is the pole of BC (Prop. 15), and AB, AC, are quadrants. Likewise C′ is a right angle, A′ is the pole of B′C′, and A′B′, A′C′, are quadrants. But A and A′ may be any angles, equal or unequal; and the sides BC and B′C′, as they will be the measures of these angles, will be accordingly either equal or unequal.

Cor. It may be here observed with respect to any spherical triangle, that if either vertex be a pole of the opposite side, the angles adjacent to this side will be right angles, the side will be the measure of the other angle, and the two remaining sides will be quadrants: thus, if A is the pole of BC in any spherical triangle ABC, the angles B and C will be right angles, AB and AC will be quadrants (Prop. 12), and BC will be the measure of the angle A (Prop. 18).

Prop. XL. (*Lemma.*)

In a right-angled spherical triangle, according as either of the sides adjacent to the right angle is greater than a quadrant, equal to it or less, the opposite angle is greater than a right angle, equal to it or less.



Let ABC be a triangle right-angled at B, and let AB be greater than a quadrant; cut off BP equal to a quadrant; then will P be the pole of the arc BC.

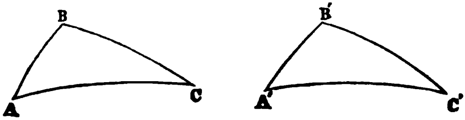
And in the triangle PBC, where the side PB is equal to a quadrant, the opposite angle PCB is a right angle (Prop. 12). In the triangle ABC, where the side AB is greater than a quadrant, the opposite angle ACB is greater than a right angle, being greater than PCB. And if the arc DC be drawn from D a point in PB, then in the triangle DBC, where DB is less than a quadrant, the opposite angle DCB is less than a right angle.

Cor. Conversely, according as either of the angles adjacent to the hypotenuse in a right-angled spherical triangle, is greater than a right angle, equal to it or less, the opposite side is greater than a quadrant, equal to it or less.

Prop. XLI.

If two right-angled spherical triangles have the hypotenuse and an adjacent angle in one equal to the hypotenuse and an adjacent angle in the other, each *to each, and these angles be not right angles, then the other adjacent angles will be equal, and the remaining sides of the one triangle equal to the corresponding sides of the other.*

Let ABC, A′B′C′, be two spherical triangles, right-angled at B and B′, and having the hypotenuse AC equal to A′C′, and the adjacent angle C equal to C′, these being oblique angles; then will the angles A and A′ be equal, and the sides AB, BC, will be equal to A′B′, B′C′, each to each.



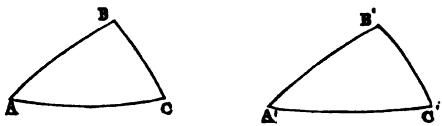
Since C and C′ are each greater or each less than a right angle, the opposite sides AB, A′B′, are each greater or each less than a quadrant. Their sum is therefore greater or less than a cemi-circumference, and they cannot be supplements of each other: but if they are not supplements of each other, they are equal (Prop. 38); and accordingly the sides BC and B′C′ are also equal, and the angle A is equal to A′ (Cor. Prop. 36).

Cor. If C and C′ are right angles, A and A′ are poles of BC and B′C′, respectively; and AB, AC, A′B′, A′C′, are quadrants; in which case, A and A′ may evidently be unequal, as also the sides BC and B′C′, which will be the measures of these angles.

PROP. XLII.

If two quadrantal triangles have the angles opposite the quadrants equal, and a second angle in one equal to a second angle in the other, and these be not right angles, the remaining angle and sides of the one will be respectively equal to those of the other.

Let ABC and A′B′C′ be two triangles in which AC and A′C′ are quadrants; and let A and B be equal to A′ and B′, A and A′ being oblique angles; then will the angle C and the sides AB, BC, be equal to the corresponding parts of the triangle A′B′C′.



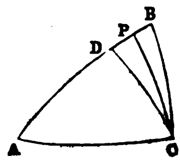
The sides BC and B′C′ are either equal or supplements of each other. If they are supplements of each other, they are quadrants, or else one is greater and the other less than a quadrant: they are not quadrants; for then would C and C′ be poles of AB and A′B′ (Prop. 14); and the angles A and A′, as also B and B′, would be right angles: nor is one greater and the other less than a quadrant; for if BC is greater than a quadrant, the angle A is greater than B (Prop. 27); then must A′ be greater than B′; and B′C′ is therefore greater than A′C′ (Prop. 26); that is, B′C′ is greater than a quadrant. Since then BC and B′C′ are not supplements of each other, they are equal; and these sides being equal, the remaining sides

AB, A′B′, will be equal, and the angle C will be equal to C′ (Cor. Prop. 36).

Schol. It was necessary to make exception of the case in which the angles A and A′ are right angles; for in that case, the poles of AB and A′B′ are in AC and A′C′ (Prop. 13), and are at C and C′; so that BC, B′C′, are quadrants, and A, B, A′, B′, are right angles; but then the angles C and C′ may evidently be unequal, and also the sides AB and A′B′, since they will be the measures of these angles.

Prop. XLIII. (*Lemma.*)

In a quadrantal triangle, according as either angle adjacent to the quadrant is greater than a right angle, equal to it or less, the opposite side is greater than a quadrant, equal to it or less.



Let ABC be a triangle of which the side AC is a quadrant: let the angle ACB be greater than a right angle, and cut off ACP equal to a right angle.

Then since AC is at right angles to CP, and is a quadrant, A must be the pole of PC; wherefore AP is a quadrant: that is, in the quadrantal triangle ACP, where C is a right angle, the opposite side is a quadrant.

In the triangle ABC, where the angle C is greater than ACP a right angle, the opposite side AB, being greater than AP, is greater than a quadrant.

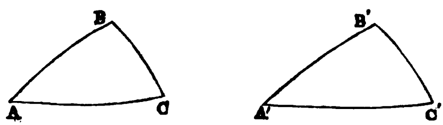
And if CD be drawn making the angle ACD less than the right angle ACP, the opposite side AD is less than the quadrant AP.

Cor. Conversely, in a triangle of which one side is a quadrant, if either of the other sides be greater than a quadrant, the opposite angle will be greater than a right angle; if equal, equal; and if less, less.

Prop. XLIV.

If two quadrantal triangles have the angles opposite the quadrants equal, and the side opposite a second angle in one triangle equal to the side opposite a second angle in the other, and these sides be not quadrants, the remaining side and angles of the one will be equal to the corresponding parts of the other.

Let ABC and A′B′C′ be two triangles of which the sides AC and A′C′ are quadrants, the angles B and B′ equal, and the side AB equal to A′B′, these sides not being quadrants; then will the side BC be equal to B′C′, and the angles A, C, equal to A′, C′, each to each.



Since AB and A′B′ are equal, and not quadrants, they are each greater or each less than a quadrant, and accordingly the opposite angles C and C′ are each greater or each less than a right angle (Prop. 43); they cannot therefore be supplements of each other; but if they are not so, they are equal (Prop. 36); and these angles being equal, the angles A and A′ are also equal, and the side BC is equal to B′C′ (Cor. Prop. 36).

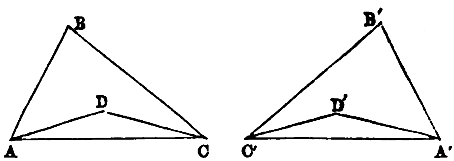
Cor. If AB and A′B′ are quadrants, A and A′ are poles of BC and B′C′ respectively, and the angles B, C, B′, C′, are right angles; in which case, the angles A and A′ may be unequal, and likewise the sides BC and B′C′, which will be the measures of these angles.

SPHERICAL SURFACES.

PROP. XLV. (*Lemma.*)

If the sides of one plane triangle be severally equal to those of another, though taken in a different order, the circles described about the triangles will be equal.

Let the sides of the triangle ABC, namely, AB, BC, AC, be equal respectively to the sides A′B′, B′C′, A′C′, of the triangle A′B′C′, then will the circles described about these triangles be equal.



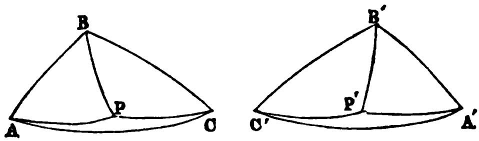
Let the centers of the circles be D and D′; and let the radii DA, DC, D′A′, D′C′, be drawn: then are ADC and A′D′C′ equal angles, for they are double the angles B and B′ at the circumference of the circles, which angles are equal, because the sides of the triangle ABC are equal to those of A′B′C′, each to each. Then in the isosceles triangles ADC, A′D′C′, the remaining angles are equal; and as the bases AC, A′C′, are equal, the other sides AD, DC, A′D′, D′C′, are equal, and are radii of equal circles.

Prop. XLVI.

If in two spherical triangles, the sides and angles of one be severally equal to those of the other, the triangles themselves are equal.

If the order in which the sides of one triangle occur, is the same as that of the corresponding sides in the other, the triangles may be proved equal, by showing that they can be made to coincide; as is done with respect to plane triangles, in the demonstration of the fourth proposition of the first book of Euclid.

But if the order of the sides in one triangle is the reverse of that of the corresponding sides in the other, the triangles cannot be made to coincide; they are however equal in this as well as in the other case.



Let ABC and A′B′C′ be two triangles on equal spheres, having the sides of the one severally equal to those of the other, but taken in a different order, AB being equal to A′B′, BC to B′C′, and CA to C′A′; these triangles will be equal.

Suppose chords to be drawn subtending the sides of the triangles: they will form two plane triangles ABC, A′B′C′, having the sides in the one severally equal to the sides in the other, but taken in the reverse order; and (Prop. 6) the planes of these triangles will intersect the spheres in small circles circumscribing the triangles, and accordingly equal. Let the centers of the spheres be denoted by O and O′, and the centers of the circles by D and D′: then OD, O′D′, will be the perpendicular distances to the circles from O and O′ (Prop. 6); and in the right-angled triangles ODA, ODB, ODC, O′D′A′, O′D′B′, O′D′C′, the sides DA, DB, DC, D′A′, D′B′, D′C′, are equal, being radii of equal circles, and OA, OB, OC, O′A′, O′B′, O′C′, are equal, being radii of equal spheres; wherefore the triangles are every way equal: the angles DOA, DOB, DOC, D′O′A′, D′O′B′, D′O′C′, then are all equal; and if OD, O′D′, be produced to meet the surfaces of the spheres in P and P′, the arcs PA, PB, PC, P′A′, P′B′, P′C′, subtending those angles, will be equal. Now the triangles APB, A′P′B′, being isosceles and mutually equilateral, can be made to coincide in the way above referred to, and are therefore equal: for the like reason BPC, APC, are equal to B′P′C′, A′P′C′, respectively: therefore the triangle ABC, which is the sum of APB, BPC, APC, is equal to A′B′C′, which is the sum of A′P′B′, B′P′C′, A′P′C′.

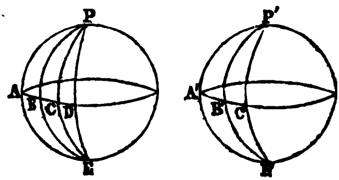
Cor. If the points P and P′ fell without the triangles ABC, A′B′C′, each of these triangles would then be equal to the sum of two of those that have their vertices at P or P′, diminished by the third; and they would accordingly be equal, as in the other case.

Def. Two triangles in which the sides of one are severally equal to those of the other, taken in the reverse order, are called *symmetrical* triangles.

Prop. XLVII.

Of two lunes on the same sphere or on equal spheres, one is to the other, as the angle between the two sides of the former to the corresponding angle of the latter.

Let APBE, A′P′B′E′, be two lunes on equal spheres; APBE is to A′P′B′E′, as the angle APB is to A′P′B′.



Let ABCD be the great circle of which P and E are poles; and A′B′C′ that whose poles are P′ and E′; take the arcs BC, CD, each equal to AB, and draw the arcs PCE, PDE, through the points C and D: then will AB, BC, CD, be the measures of the opposite angles at P and E (Prop. 18), angles contained by the sides of the lunes APBE, BPCE, CPDE: these angles of the lunes are therefore equal. Now it may be shown that these lunes having equal angles, can be applied to each other so as to coincide: they are therefore equal; and the whole lune APDE is the same multiple of APBE, that the angle APD is of APB. In like manner it is seen that if A′B′, B′C′, are equal arcs, A′P′B′E′ and B′P′C′E′ are equal lunes, and that the lune A′P′C′E′ is the same multiple of A′P′B′E′, that the angle A′P′C′ is of A′P′B′. Moreover, if the angle APD is equal to A′P′C′, the lune APDE is equal to A′P′C′E′; if greater, greater; and if less, less. And it is evident that the like is true for any multiples of AB and A′B′ other than AD and A′C′. Since then it has been shown concerning four magnitudes, namely, the angles APB, A′P′B′, and the lunes APBE, A′P′B′E′, that when any equimultiples are taken of the first and third, and any of the second and fourth, if the multiple of the first is equal to that of the second, the multiple of the third is equal to that of the fourth; if greater, greater; and if less, less; therefore, the first is to the second as the third to the fourth; that is, APB is to A′P′B′, as APBE to A′P′B′E′.

Cor. The lune APBE is to the whole spherical surface as the angle APB is to four right angles, or as the arc AB is to the circumference of a great circle.

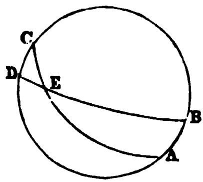
If the angle APB be a right angle, the lune APBE will be one fourth of the whole surface of the sphere, and the triangle APB, which is evidently half the lune, will be one eighth of the spherical surface. In this case the arc AB is a quadrant, as well as the arcs PA, PB, and all the angles of the triangle are right angles. Such a triangle is called a *tri-quadrantal* triangle.

Schol. Of two spherical wedges in the same sphere or in equal spheres, one is to the other as the angle between the two plane sides of the former to the corresponding angle of the latter: which is obvious from the demonstration given above with respect to two lunes.

Prop. XLVIII.

If the circumferences of two great circles intersect on the convex surface of a hemisphere, the opposite trian-

gles thus formed, will be equal to the lune whose spherical angles are equal to the angles of these triangles at the point of intersection.

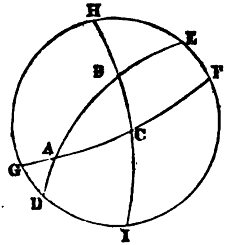


Let the great circle ABCD be the base of a hemisphere, on the convex surface of which are drawn the two arcs of great circles AC, BD, intersecting in E; the triangles AEB and CED are together equal to the lune whose angles are equal to the angles AEB, CED.

For the triangle which is wanting to complete the lune of which AEB is a part, is one in which AB is one side, the supplements of EA and EB are the other two sides, and an angle equal to E is the angle contained by these two sides. Now DCE is equal to this triangle, two sides and the included angle of the one being equal to two sides and the included angle of the other, each to each; since CE and DE are supplements of AE and BE, and the angle CED is equal to AEB. Since then CDE is equal to the triangle that is wanting to complete the lune of which ABE is a part, the two triangles CED and AEB are together equal to this lune, which is a lune having for one of its angles an angle of the triangle AEB at the point of intersection E.

Prop. XLIX.

The excess of the sum of the angles of a spherical triangle above two right angles, is to a right angle, as the triangle is to a tri-quadrantal triangle.



Let ABC be any spherical triangle; the excess of the sum of the angles A, B, and C above two right angles, is to a right angle, as ABC is to a tri-quadrantal triangle

Produce the sides AC and AB, but not so far as to make them intersect, and through E and F in the parts produced, draw the circumference of a great circle EFGH; and let BA and CA produced meet this circumference again in D and G; also let BC produced meet it in H and I.

Then, by the two preceding propositions, the angle EAF, which is the angle A in the triangle BAC, is to four right angles, as the sum of the triangles EAF and DAG is to the spherical surface or eight tri-quadrantal triangles. If R denote a right angle and T a tri-quadrantal triangle, this proportion may be thus expressed,

$$A : 4R :: EAF + DAG : 8T;$$

$$\text{or, } A : R :: EAF + DAG : 2T.$$

To this may be added these two other like proportions;

$$B : R :: DBI + EBH : 2T,$$

$$C : R :: FCI + GCH : 2T.$$

Hence (Euc. 5, 24),

$$A+B+C : R :: EAF+DAG+DBI+EBH+FCI+GCH : 2T$$

Now by inspecting the figure it is seen that the third term of this proportion exceeds the convex surface of the hemisphere whose base is the circle EFGH, by twice the triangle ABC. It may therefore be replaced by the expression 4T+2ABC. The proportion then will be

$$A+B+C : R :: 4T+2ABC : 2T,$$

or

$$A+B+C : 2R :: 2T+ABC : 2T.$$

Hence (Euc. 5, 17),

$$A+B+C-2R : 2R :: ABC : 2T,$$

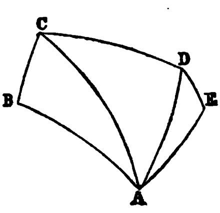
or

$$A+B+C-2R : R :: ABC : T;$$

that is, the excess of the sum of the angles of the triangle ABC above two right angles, is to a right angle, as the triangle itself is to a tri-quadrantal triangle; which was to be proved.

Prop. L.

If the sum of the angles of a spherical polygon together with four right angles be diminished by twice as many right angles as there are sides of the polygon, the remainder will be to a right angle as the polygon is to a tri-quadrantal triangle.



Let ABCDE be a spherical polygon: from A draw arcs to each of the other angular points of the polygon, except the adjacent ones, B and E; then will the polygon be divided into as many triangles, wanting two, as it has sides.

Now by the last proposition, the sum of the angles of each triangle diminished by two right angles is to a right angle as the triangle is to a tri-quadrantal triangle. Hence (Euc. 5, 24), as the sum of all the angles of all the triangles diminished by twice as many right angles as there are triangles is to a right angle, so is the sum of all the triangles to a tri-quadrantal triangle. But the sum of all the triangles, is the polygon ABCDE, and the sum of the angles of the triangles is the same as that of the angles of the polygon. Therefore, the sum of the angles of the polygon diminished by twice as many right angles as there are triangles, is to a right angle, as the polygon is to a tri-quadrantal triangle. Now if the sum of the angles of the polygon, instead of being diminished by twice as many right angles as there are triangles, be first increased by four right angles, and then diminished by twice as many right angles as there are sides of the polygon, the result will not be changed. Then, the sum of the angles of a spherical polygon together with four right angles when diminished by twice as many right angles as there are sides of the polygon, is to a right angle, as the polygon is to a tri-quadrantal triangle.

SPHERICAL TRIGONOMETRY.

ART. 1. If two of the sides and angles of a spherical triangle be either two right angles, two quadrants, or a right angle and a quadrant adjacent or opposite, then is one vertex of the triangle a pole of the opposite side, and the angle at the vertex is measured by this side; the other two sides are quadrants, and the opposite angles right angles: and to be completely determined, such a triangle must have given its oblique angle or the opposite side which is a measure of this angle. These parts can neither of them be computed from the other parts of the triangle.

Triangles of this kind may be called *bi-quadrantal*, and in what follows they are left out of view.

Among such triangles are to be classed those that are tri-quadrantal.

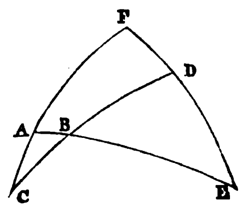
Of the sides and angles of a spherical triangle not bi-quadrantal, if any three be given, the other three may *in general* be determined from them. The cases in which this cannot be done, will be duly noticed hereafter.

It is the business of Spherical Trigonometry to furnish methods of computing the unknown from the known parts of a spherical triangle.

Spherical as well as Plane Trigonometry is divided into two branches; one relating to right and the other to oblique-angled triangles. It is convenient to treat first of triangles that are right-angled.

RIGHT-ANGLED SPHERICAL TRIANGLES.

2. If from the vertex of one of the oblique angles in a right-angled spherical triangle, as a pole, an arc of a great circle be drawn, meeting the sides that contain the other oblique angle (produced beyond the angular point if necessary), a second triangle will thus be formed, having one of its vertices at this angular point; which triangle is said to be *complemental* to the former triangle.



Thus, if ABC be a triangle right-angled at A, and from the vertex C as a pole an arc of a great circle DE be drawn, meeting AB and CB produced, the triangle DBE thus formed, having one of its vertices at B, is said to be complemental to ABC.

If from the vertex B as a pole, an arc of a great circle be drawn, meeting AC and BC produced, another triangle will be formed complemental to ABC. And thus every spherical triangle has two complemental triangles.

3. If one triangle be complemental to another, the latter is complemental to the former.

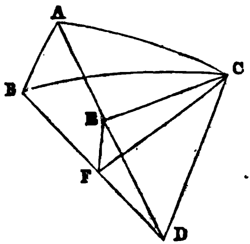
For in the preceding figure, the triangle BDE is right-angled at D, since C is the pole of DE: and E is the pole of AC; for the arcs CA and ED meet at right angles (Geom. Prop. 12); and as EF and EA are arcs at right angles to AC, E is a pole of AC (Geom. Prop. 15). Since then from E the vertex of an oblique angle in the right-angled triangle BDE, as a pole, the arc of a great circle AC is drawn, meeting the sides DB and EB produced, the triangle ABC thus formed, having one of its vertices at B, is complemental to BDE.

4. Of two triangles complemental to each other, an oblique angle in one is equal to an oblique angle in the other; the opposite side in each is the complement of the measure of the second oblique angle in the other; and of the sides containing the equal angles, the hypotenuse in either triangle is the complement of the other side in the other triangle.

Thus, of the complemental triangles ABC, DBE, in the preceding figure, the oblique angles at B are equal; and since CF is a quadrant, AC is the complement of AF, which is evidently the measure of the angle E in BDE; and ED is the complement of DF, which is the measure of the angle C in ABC. And of the sides containing the equal angles at B, the hypotenuse BC is the complement of BD, and the hypotenuse BE the complement of AB.

5. *In a right-angled spherical triangle, the sine of the hypotenuse is to the radius, as the sine of another side is to the sine of the opposite angle.*

Let ABC be a spherical triangle, right-angled at A; then the sine of BC the hypotenuse, is to the radius, as the sine of the side AC is to the sine of the opposite angle ABC.



Let D be the center of the sphere; draw the radii AB, BD, CD; draw CE, CF, at right angles to AD, BD, and join EF; then will EF be perpendicular to BD. For since BAC is a right angle, the plane ACD is perpendicular to ABD; and as CE is at right angles to the common section AD, CE must be at right angles to the plane ABD, and to the line EF which it meets in that plane. Now since CEF is a right angle, $CE^2+EF^2=CF^2$; and since CFD is a right angle, $CF^2+FD^2=CD^2$; therefore $CE^2+EF^2+FD^2=CD^2$: again, since CED is a right angle, $CE^2+ED^2=CD^2$: hence, $CE^2+EF^2+FD^2=CE^2+ED^2$; and therefore $EF^2+FD^2=ED^2$: consequently EFD is a right angle (Euc. 1, 48). Since then CF and EF are drawn in the planes BCD, BAD, at right angles to their common section BD, the angle EFC is the angle of inclination between the planes, and is accordingly equal to the spherical angle ABC. Now in the plane triangle CEF right-angled at E, CF is to the radius as CE is to the sine of EFC; but CF is the sine of the arc CB, and CE the sine of CA, and the angle EFC is equal to ABC; therefore the sine of BC is to the radius as the sine of AC is to the sine of the angle ABC. In like manner it may be shown that the sine of BC is to the radius, as the sine of AB is to the sine of ACB.

Cor. Let the angles of the triangle ABC right-angled at A, be denoted by A, B, C, and the opposite sides (or the angles measured by them) by a, b, c, respectively; and let the corresponding parts of a triangle complemental to ABC, be denoted by A′, B′, C′, a', b', c', the angle A′ being a right angle. Then (Art. 4), if B′=B; C′=comp. b, b'=comp. C, a'=comp. c, c'= comp. a: and if C′=C; B′=comp. c, c'=comp. B, a'= comp. b, b'=comp. a.

Now by the preceding proposition,

sin a : R : : sin b : sin B; hence if unity be put for R,

$$\text{(1)} \quad \sin b = \sin a \sin B.$$

Likewise, since the same equation holds good in the complemental triangles,

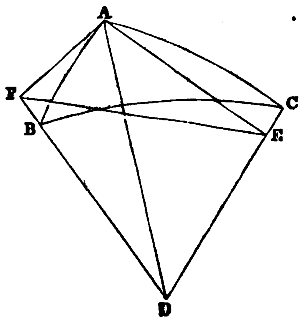
$$\sin b' = \sin a' \sin B';$$

from which equation are derived the two following; one for the case in which B′=B, the other for that in which C′=C;

$$\sin \text{comp. } C = \sin \text{comp. } c \sin B,$$
$$\sin \text{comp. } a = \sin \text{comp. } b \sin \text{comp. } c;$$

$$\text{or} \quad \text{(2)} \quad \cos C = \cos c \sin B,$$
$$\text{(3)} \quad \cos a = \cos b \cos c.$$

6. *In a right-angled spherical triangle, the sine of either of the sides containing the right angle is to the radius as the tangent of the other of these sides is to the tangent of the opposite angle.*



Let ABC be a triangle right-angled at A, the sine of AC is to the radius as the tangent of AB is to the tangent of the opposite angle ACB.

Let D be the center of the sphere; join DA, DB, DC, draw AE perpendicular to CD; and AF a tangent to AB, meeting DB produced in F; also draw EF; this will be perpendicular to CD.

For since AF is a tangent to AB, it is perpendicular to AD; and being drawn in the plane BAD at right angles to AD the intersection of this plane with the plane CAD, it must be perpendicular to the latter plane, and perpendicular to AE in that plane. Then in the right-angled triangle FAD, $FD^2=AF^2+AD^2$; and in the triangle AED, $AD^2=AE^2+ED^2$; therefore $FD^2=AF^2+AE^2+ED^2$; but $AF^2+AE^2=FE^2$, because FAE is a right angle; hence $FD^2=FE^2+ED^2$; and consequently FED is a right angle (Euc. 1, 48). Since then from E a point in CD the common section of the planes ACD, BCD, the lines EA, EF, are drawn in the two planes at right angles to CD, AEF is the angle of inclination between the planes, and is equal to the spherical angle ACB. Now in the plane triangle AEF right-angled at A, AE is to the radius as AF is to the tangent of the angle AEF; or since AE is the sine of AC, and AF the tangent of AB, the sine of AC is to the radius as the tangent of AB is to the tangent of ACB. In like manner it may be shown that the sine of AB is to the radius as the tangent of AC is to the tangent of ABC.

Cor. Hence (if the quantities concerned be denoted as in the corollary to the last proposition), are derived the following equations:

(1) $\tan c = \sin b \tan C$

$\tan c' = \sin b' \tan C'$

(2) $\cot a = \cos C \cot b$

(3) $\cot B = \cos a \tan C.$

7. If of the sides and angles in a right-angled triangle, any two besides the right angle be given, the remaining three may in general be determined by means of the equations numbered (1), (2), (3), in the two preceding corollaries. And as in each computation there are three parts concerned, two given and one sought, the following six are all the cases that can occur: namely, those in which the three parts concerned are

(1) The hypotenuse and the two oblique angles,

(2) The three sides,

(3) The hypotenuse, another side, and the included angle,

(4) The hypotenuse, another side, and the opposite angle,

(5) The two perpendicular sides and one of the oblique angles,

(6) The two oblique angles and a side opposite one of them.

The equations belonging to these cases are, when placed in the corresponding order, as follows:

(1) $\cot B = \cos a \tan C.$

(2) $\cos a = \cos b \cos c$

(3) $\cot a = \cos C \cot b$

(4) $\sin b = \sin a \sin B$

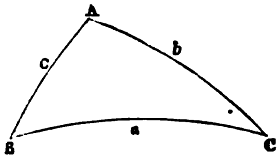
(5) $\tan c = \sin b \tan C$

(6) $\cos C = \cos c \sin B$

To these must be added four others; for it is evident that there are two equations belonging to each of the last four cases, one of which is to be deduced from the other by changing B into C, b into c, and *vice versa.* These additional equations are the following,

$$(7)\ \cot a = \cos B \cot c$$
$$(8)\ \sin c = \sin a \sin C$$
$$(9)\ \tan b = \sin c \tan B$$
$$(10)\ \cos B = \cos b \sin C.$$

The relations of the quantities concerned in the equations may be seen more clearly by aid of the accompanying figure.



8. The ten equations of the six cases of right-angled spherical trigonometry may be all embraced in one or two simple rules, called from their inventor, Napier's Rules of the Circular Parts.

The parts of a triangle concerned in the equations are the three sides and two oblique angles of a right-angled spherical triangle. By the *circular parts*, however, are meant the two perpendicular sides, and the *complements* of the hypotenuse and the two oblique angles, instead of these three quantities themselves; the use of those complements having been found requisite in order to give rules of the simplest form.

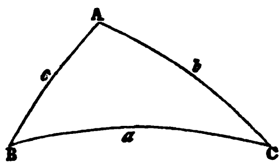
Supposing the hypotenuse and the two oblique angles of a triangle to be replaced by their complements, and the right angle to be left out of view, then of the five circular parts in the triangle, if either be taken for the *middle* part, the two that are next to this and separated by it are called the *adjacent* parts, and the other two the *opposite* parts.

Napier's rules of the circular parts thus distinguished, are as follows:

The SINE *of the* MIDDLE *part, equals the product of the* TANGENTS *of the* ADJACENT *parts, and equals the product of the* COSINES *of the* OPPOSITE *parts.*

This proposition may be more easily remembered, when it is observed that the two prominent words in each of the three parts of which it consists, have their first vowels alike; this vowel being *i* in each of the words *sine* and *middle*, being *a* in the words *tangent* and *adjacent*, and *o* in the words *cosine* and *opposite.*

9. As each of the five circular parts may be taken for the middle one, and the proposition gives two equations for each case, it will furnish in all *ten* equations; and these will be found to agree with the ten previously stated.



Thus if the parts of a right-angled triangle be represented as in the figure here given, where A is the right angle; and each of the five circular parts, namely, b, c, comp. B, comp. a, comp. C, be in turn made the middle part, there will result the following equations,

$\sin b = \tan c \tan \text{comp. } C$, that is,

(1) $\sin b = \tan c \cot C$,

and $\sin b = \cos \text{comp. } B \cos \text{comp. } a$, that is,

(2) $\sin b = \sin B \sin a$ (for the cosine of the complement of an arc or angle is the sine of that arc or angle).

Also (3) $\sin c = \tan b \cot B$

(4) $\sin c = \sin a \sin C$

(5) $\cos B = \tan c \cot a$

$$(6)\quad \cos B = \cos b \sin C$$
$$(7)\quad \cos a = \cot B \cot C$$
$$(8)\quad \cos a = \cos b \cos c$$
$$(9)\quad \cos C = \tan b \cot a$$
$$(10)\quad \cos C = \cos c \sin B,$$

which differ only a little in form from those given in the preceding article, and may be made identical with them by applying, in a few cases, the principle that the tangent of an arc or angle equals the reciprocal of the cotangent, when radius is unity (Day's Trig., Art. 93). For example, the equation (1) of the seventh article may be stated thus, $\cos a = \frac{\cot B}{\tan C}$; and as $\cot C = \frac{1}{\tan C}$, $\cos a = \cot B \cot C$; which is the seventh of the preceding equations.

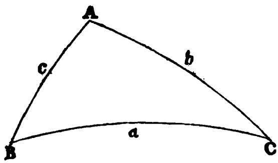
10. In applying Napier's theorem to any proposed case, it is best at first to make no distinction between the two parts given and the part required, but consider only which of the three circular parts concerned, must be made the middle part, in order that the other two may be either both adjacent or both opposite, and not one adjacent and the other opposite; and then by means of the theorem form an equation involving these three parts; afterwards reducing the equation, if necessary, so as to have the given parts on one side, and the part required on the other. Thus, if the sides a and c be given and the angle C be sought, the three circular parts concerned are c, comp. a, comp. C, of which it will be seen that c must be taken for the middle part, and the other two for opposite parts. By Napier's rule, the equation connecting these parts is the following,

$$\sin c = \sin a \sin C.$$

And if the quantities given were a and C, this equation would require no reduction: but as C and not c is the quantity required, the equation must be reduced to the form

$$\sin C = \frac{\sin c}{\sin a};$$ and then by logarithmic calculation the value of C may readily be found.

11. One must first of all, however, gain some facility in stating, when any circular part of a right-angled triangle is taken for the middle one, which of the other four are adjacent parts, and which opposite. The statements that may be demanded are these;



(1) If comp. a be the middle part, the adjacent parts are comp. B, comp. C, and the opposite parts are b, c.

(2) If b is the middle part, the adjacent parts are c, comp. C, and the opposite parts are comp. a, comp. B.

(3) If c is the middle part, the adjacent parts are b, comp. B, and the opposite parts are comp. a, comp. C.

(4) If comp. B is the middle part, the adjacent parts are comp. a, c, and the opposite parts are b, comp. C.

(5) If comp. C is the middle part, the adjacent parts are comp. a, b, and the opposite parts are c, comp. B.

12. Now let it be required, with any three proposed circular parts (without regard to any distinction between them as known or unknown quantities), to state which must be taken for the middle part, in order that the other two may be either both adjacent or both opposite; and let it not be yet required to proceed to the use of Napier's rule in forming any equation. The several cases that will thus be presented, may be stated as follows:

(1) If the three quantities concerned are b, c, C, then b is the middle part, and the parts c, comp. C, are adjacent.

(2) If the three proposed quantities are a, b, B, then b is the middle part, and the parts comp. a, comp. B, are opposite.

(3) If the three quantities are b, c, B, then c is the middle part, and the parts b, comp. B, are adjacent.

(4) If the three quantities are a, c, C, then is c the middle part, and comp. a, comp. C, are opposite parts.

(5) If the three quantities are a, c, B, then is comp. B the middle part, and the parts comp. a, c, are adjacent.

(6) If b, B, C, are the three quantities, comp. B is the middle part, and the parts b, comp. C, are opposite.

(7) If a, B, C, are the three quantities, comp. a is the middle part, and the parts comp. B, comp. C, are adjacent.

(8) If a, b, c, are the three quantities, comp. a is the middle part, and the parts b, c, are opposite.

(9) If a, b, C, are the three quantities, comp. C is the middle part, and the parts comp. a, b, are adjacent.

(10) If c, B, C, are the three quantities, comp. C is the middle part, and the parts c, comp. B, are opposite.

13. The use of Napier's rules will now be shown by some examples.

(1) Given the sides a and b; to find c.

The circular parts here concerned are comp. a, b, c: of these comp. a is the middle part, and the other two, b, c, are opposite parts. Then by Napier's rule,

$$\sin \text{comp. } a = \cos b \cos c, \qquad \text{that is,}$$

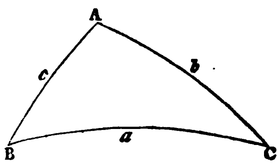
$\cos a = \cos b \cos c$; and since c is the required quantity, $\cos c = \frac{\cos a}{\cos b}$; whence by logarithmic calculation, the value of c may be obtained.

Schol. Instead of dividing by cos b, we may multiply by sec b, since the cosine and secant are reciprocals of each other, when radius equals unity (Day's Trig., Art. 93). The last equation may then be replaced by this, $\cos c = \cos a \sec b$.

In general, whenever a cosine occurs as a divisor, it may be replaced by the corresponding secant as a multiplier; likewise when a sine occurs as a divisor, it may be replaced by the corresponding cosecant as a multiplier. In this way too may tangents be exchanged for cotangents, and cotangents for tangents, since they are reciprocals of each other.

(2) Given $a = 84°$, $B = 63°$; to find C.

Here the three circular parts are comp. a, comp. B, comp. C; of which comp. a is the middle part, and comp. B, comp. C, are adjacent parts: then,



$$\sin \text{comp. } a = \tan \text{comp. B} \tan \text{comp. C},$$

that is, $$\cos a = \cot B \cot C;$$

whence $$\cot C = \frac{\cos a}{\cot B},$$

or $$\cot C = \cos a \tan B.$$

Now, as $a = 84°$, $\cos a = \ \ 9.0192346$

also, $\tan B = 10.2928341$

hence $\cot C = \ \ 9.3120687$, the logarithmic radius 10 being rejected:

therefore, $C = 78° \ 24' \ 24''$.

(3) Given $b=37°\ 13'$, $B=43°\ 41'$; to find C.

Here comp. B is the middle part, and the other parts, b, comp. C, are opposite.

Then, sin comp. B=cos b cos comp. C,

that is, cos B=cos b sin C;

whence, $\sin C=\frac{\cos B}{\cos b}=\cos B \sec b$.

Now, cos B= 9.8592393

sec b=10.0988938

therefore sin C= 9.9581331

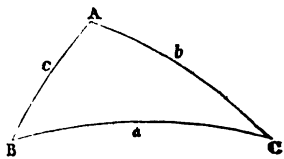
and C= 65° 14′ 38″,

or C=114° 45′ 22″,

these two angles having the same sine, because the sine of any angle is the same with that of its supplement.

This is one of those cases in which two parts besides the right angle of a right-angled triangle are insufficient to determine another part.

(4) Given $a=113°\ 40'$, $b=60°\ 27'$; to find C.



Here comp. C is the middle part; and the other parts, comp. a, b, are adjacent.

Then, cos C=cot a tan b:

but, cot a= 9.6417473

and tan b=10.2464741

therefore, cos C= 9.8882214

Now as a is an obtuse angle, its cotangent is negative (Day's Trig., Art. 201), and as tan b is positive, the product of the two is negative; therefore cos C is negative, and C must accordingly be an obtuse angle.

The obtuse angle answering to the number 9.8882214 in the table of cosines is 140° 37′ 51″, which is therefore the required angle C.

(5) Given a, B; to find the other parts of the triangle.

First, to find b, take b for the middle part; then comp. a, comp. B are opposite parts, and

$$\sin b = \sin a \sin B;$$

whence b may be calculated.

Next, to find c, take comp. B for the middle part; then comp. a and c are adjacent parts; and

$$\cos B = \cot a \tan c,$$

whence $$\tan c = \cos B \tan a.$$

Lastly, to find C, take comp. a for the middle part; in which case, comp. B and comp. C are adjacent parts: then $$\cos a = \cot B \cot C;$$

whence $$\cot C = \cos a \tan B.$$

If $a = 72° 21' 40''$, and $B = 14° 10' 25''$, in the above equations; then $b = 13° 29' 40''$, $c = 71° 50' 35''$, and $C = 85° 37' 26''$.

OBLIQUE-ANGLED TRIANGLES.

14. Of the sides and angles in an oblique-angled triangle, it is necessary, as has before been stated, that three be given, in order that the triangle may be determined. These may be

(1) Two sides and the included angle,

(2) Two sides and an angle opposite one of them,

(3) Two angles and the intermediate side,

(4) Two angles and a side opposite one of them,

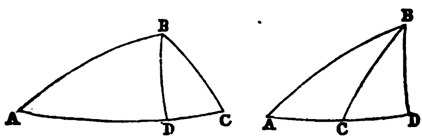
(5) The three angles,

(6) The three sides.

In the last case, every two sides are separated by an unknown angle; and in the last but one, every two angles have between them an unknown side. But in each of the other four cases, there are two at least of the given parts, which are not separated by an unknown part, but are adjacent to each other.

Now every oblique-angled triangle may be divided into two right-angled triangles by a perpendicular drawn from one of the angles to the opposite side. And this may be so done as to make the two given parts that are adjacent in each of the first four of the above cases, fall in the same right-angled triangle. Then all the other parts of this right-angled triangle may be determined by Napier's rules: and thus means will be furnished for computing the unknown parts of the other right-angled triangle; whence all the parts of the oblique-angled triangle will be known. This will be seen more clearly by attending to these cases separately, as may be conveniently done after noticing the following theorem.

15. *The sines of the sides of a spherical triangle are as the sines of their opposite angles.*



Let ABC be a spherical angle, the sine of AB is to the sine of BC as the sine of the angle ACB is to the sine of BAC.

From B draw BD perpendicular to AC or AC produced, forming the right-angled triangles ABD, CBD :

then (Art. 5), sin BC : R : : sin BD : sin BCD,
and (Euc. 5, A), R : sin BC : : sin BCD : sin BD :
also sin AB : R : : sin BD : sin BAD :
therefore (Euc. 5, 23),

sin AB : sin BC : : sin BCD : sin BAD.

Now the sines of BAC and ACB may be substituted for those of BAD and BCD, since the angles ACB and BCD, if they are not the same, are supplements of each other.

Therefore sin AB : sin BC : : sin ACB : sin BAC.
In like manner it may be shown that

sin AC : sin AB : : sin ABC : sin ACB;

wherefore also sin AC : sin BC : : sin ABC : sin BAC.

16. *Given two sides and the included angle of a spherical triangle; to find the other side and angles.*

In the spherical triangle ABC, let the two sides AB, AC, and the angle A be given, from which to determine the side BC and the angles ABC, ACB.

Draw BD from B perpendicular to AC, forming two right-angled triangles ABD, BDC, such that one of them ABD has for two of its parts, two of the quantities given, namely, the side AB and the angle A. From these two, may be computed by Napier's rules, the sides AD, BD, and the angle ABD. Then DC is found by taking AD from the given side AC. And in the right-angled triangle BDC, having now given the sides BD and DC, the side BC, the angle C, and the angle CBD may be computed. The angle ABC is found by adding together the angles ABD, CBD.

If the perpendicular BD meets AC produced, the method of solution is the same as has been stated, except that DC is found by subtracting AC from AD, and the angle ABC by subtracting CBD from ABD.

The above is not, however, the most concise method of obtaining the quantities required. It is sufficient in the first triangle to compute from the given quantities BA and A, the value of AD, from which DC is directly obtained. Then by Napier's theorem, equations may be formed in which the hypotenuse AB of the triangle ADB shall be expressed in terms of the two sides AD, BD; and the hypotenuse BC of the triangle BCD in terms of BD and DC; and by eliminating BD from these equations, a new equation may be obtained, involving the quantities AB, BC, AD, DC, from which the value of BC may be found in terms of the other three quantities which are known. Thus, by Napier's theorem, if comp. AB be the middle part, $\cos AB = \cos AD \cos BD$; likewise, $\cos BC = \cos DC \cos BD$; and if the members of these equations be multiplied together crosswise, and the products be divided by cos BD, there will result the equation

$$\cos BC \cos AD = \cos AB \cos DC;$$

whence $\cos BC = \cos AB \cos CD \div \cos AD$,

from which the value of BC may be computed.

In like manner may be found the equation

$$\tan C = \tan A \sin AD \div \sin DC,$$

from which the value of C can be computed.

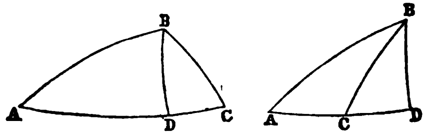
After finding either of the quantities BC or C, all the remaining unknown quantities in the triangle ABC may be found by means of the preceding theorem.

Thus, when BC is known, the angles ACB and ABC are found from the proportions (Art. 15),

sin BC : sin AC : : sin A : sin ABC
sin BC : sin AB : : sin A : sin ACB.

Or if C be known instead of BC, then the proportions to be used in finding BC and ABC, are

sin C : sin A : : sin AB : sin BC,
sin AB : sin AC : : sin C : sin ABC.



Example. Let AB=48° 27′ 29″, AC=78° 23′ 40″, A=39° 40′ 30″; then in the equation

tan AD=cos A tan AB,
cos A= 9.8863092
tan AB=10.0525128

tan AD= 9.9388220 (the index being diminished by 10, because each number in the table of logarithmic signs, tangents, &c., is 10 too great). Hence AD=40° 58′ 40″, and CD=AC−AD=37° 25′. Again, in the equation

cos BC=cos AB cos CD÷cos AD
=cos AB cos CD sec AD,

cos AB= 9.8216451
cos CD= 9.8999506
sec AD=10.1220738

cos BC= 9.8436695 (the index being decreased by 20);

therefore, BC=45° 45′ 26″

Next, in the equation

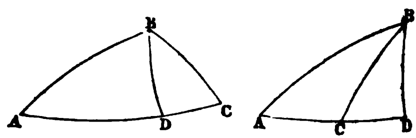
$$\sin ABC = \sin A \sin AC \div \sin BC$$
$$= \sin A \sin AC \operatorname{cosec} BC,$$

$$\begin{aligned} \sin A &= 9.8051146 \\ \sin AC &= 9.9910292 \\ \operatorname{cosec} BC &= 10.1448505 \\ \hline \sin B &= 9.9409943 \\ \hline B &= 119^\circ\ 11'\ 44'' \end{aligned}$$

It will be found in like manner. that $C = 41^\circ\ 50'\ 8''$, by means of the equation, $\sin C = \sin A \sin AB \operatorname{cosec} BC$; or C may be found from the equation $\cos C = \cot BC \tan CD$.

17. *Given in a spherical triangle, two sides and an angle opposite one of them; to find the other parts.*

Let ABC be a spherical triangle in which the sides AB, BC, and the angle A are given, and the angles B, C, and the side AC are required.



From B one end of the given side AB, and opposite the given angle A, draw the perpendicular BD to the side AC or AC produced: then in the right-angled triangle ABD will be given the hypotenuse AB and the angle A, from which all the other parts of the triangle may be found: and then in the triangle BDC will be given BD and BC, from which all the other parts of this triangle may be determined; and thus all the required parts of the triangle ABC will be ascertained. But it will be better to obtain these parts in the following manner:

From the given parts AB, BC, and A, find C by means of the theorem that the sines of the sides of a spherical triangle are as the sines of the opposite angles. Then in the right-angled triangle ABD, find AD from the given parts AB and A by Napier's rule; and in the triangle BDC, find CD from the given parts BC and C. Thus the side AC, being the sum or difference of AD and CD, will be known; and from this with either of the other sides and its opposite angle, may be found the angle ABC, by means of the fore-mentioned theorem concerning the sines of the sides and opposite angles of a triangle. The proportions and equations to be used are these,

$$\sin BC : \sin AB :: \sin A : \sin C,$$

$$\cos A = \cot AB \tan AD, \text{ or } \tan AD = \cos A \tan AB,$$

$$\cos C = \cot BC \tan CD, \text{ or } \tan CD = \cos C \tan BC,$$

$$\sin BC : \sin AC :: \sin A : \sin ABC.$$

Example. Let AB=86° 14′ 20″, BC=36° 26′, A=14° 12′ 50″; then in the proportion

$$\sin BC : \sin AB :: \sin A : \sin C,$$

or the equation $\sin C = \sin A \sin AB \div \sin BC$
$= \sin A \sin AB \operatorname{cosec} BC,$

$$\begin{aligned} \sin A &= 9.3901265 \\ \sin AB &= 9.9990636 \\ \operatorname{cosec} BC &= 10.2262961 \\ \hline \sin C &= 9.6154862 \\ \hline C &= 24^\circ\ 21'\ 57'' \\ \hline \end{aligned}$$

Again, in the equation

$$\tan AD = \cos A \tan AB,$$

$$\begin{aligned} \cos A &= 9.9864967 \\ \tan AB &= 11.1821820 \\ \hline \tan AD &= 11.1686787 \\ \hline AD &= 86^\circ\ 7'\ 14''. \end{aligned}$$

And in the equation

$$\tan CD = \cos C \tan BC,$$

$$\begin{aligned} \cos C &= 9.9594850 \\ \tan BC &= 9.8681517 \\ \hline \tan CD &= 9.8276367 \\ \hline CD &= 33^\circ\ 55'\ 3''. \end{aligned}$$

Hence $\quad AC = AD + CD = 120^\circ\ 2'\ 17''$:

and lastly in the equation

$$\sin ABC = \sin A \sin AC \div \sin BC$$
$$= \sin A \sin AC \operatorname{cosec} BC,$$

$$\begin{aligned} \sin A &= 9.3901265 \\ \sin AC &= 9.9373640 \\ \operatorname{cosec} BC &= 10.2262961 \\ \hline \sin ABC &= 9.5537866 \\ \hline ABC &= 159^\circ\ 1'\ 39''. \end{aligned}$$

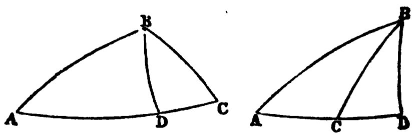
As the side AC is greater than AB, the angle ABC must be greater than the angle C; and therefore the obtuse angle 159° 1′ 39″ is given for its value, instead of the supplementary acute angle 20° 58′ 21″.

The sine of C above determined, answers in the table of sines not only to the acute angle 24° 21′ 57″, but also to the supplement of this, namely, 155° 38′ 3″. If this obtuse angle were taken for the value of C, then AC would be the difference instead of the sum of AD and CD, and the value of ABC would be accordingly altered.

This is one of those cases alluded to in Art. 1, where three parts given are not sufficient to determine a triangle.

18. *Given two angles and the intermediate side of a spherical triangle; to find the remaining angle and sides.*

Let ABC be a spherical triangle, in which the angles A and B and the side AB are given, and the angle C and the sides AC and BC are required.



Draw BD perpendicular to AC, forming the right-angled triangles ABD and BCD, in the former of which are the two given parts AB and A; then by Napier's rules, the angle ABD and the sides BD, AD, may be determined; and thus in the triangle BCD, will be known the side BD and the angle CBD, since this is the difference between ABC and ABD: hence the other parts of this triangle may be computed; and then all the parts of the triangle ABC will be known.

But these may be found by a shorter process, thus: the angle ABD may be first obtained, by means of Napier's rule, from the given quantities A and AB; then CBD can be found by taking the difference of ABC and ABD. And if now BD be expressed in terms of AB and

ABD according to Napier's rule, and likewise in terms of BC and CBD, an equation may be formed of the two expressions, which will involve the four quantities ABD, CBD, AB and BC, the last of which may accordingly be expressed in terms of the other three, which are all known. When BC has been thus determined, the remaining unknown parts, namely, the angle C and the side AC may be found, by the theorem of Art. 15; or the angle C may be found, from BC and CBD by Napier's rule.

The equations to be used are the following,

$$\cot ABD = \cos AB \tan A$$

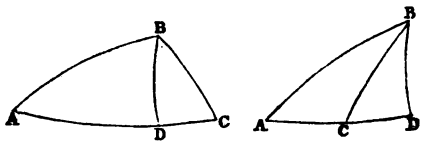
$$\tan BC \; (= \tan AB \cos ABD \div \cos CBD) = \tan AB \cos ABD \sec CBD$$

$$\cot C = \cos BC \tan CBD$$

$$\sin AC \; (= \sin BC \sin ABC \div \sin A) = \sin BC \sin ABC \operatorname{cosec} A.$$

Example. Let AB = 77° 35′ 50″, A = 17° 28″, B = 25° 52′; then will ABD = 86° 8′ 1″, DBC = 60° 16′ 1″, BC = 31° 43′ 32″, C = 145° 7′ 8″, AC = 49° 50′ 53″.

19. *Given in a spherical triangle two angles and a side opposite one of them; to find the remaining angle and sides.*



Let ABC be a spherical triangle in which the parts A, C and AB are given: it is required to find the other parts B, AC and BC.

Draw the perpendicular BD: then in the triangle ABD, there are given the angle A and the side AB, from which all the other parts may be found; and thus there will be given in the triangle CBD the side BD and the angle C; whence the other parts of this triangle may be determined; and then all the parts of the triangle ABC will be known.

But these parts may be more conveniently computed. The side BC may be determined from the given quantities A, C, and AB, by means of the theorem that the sines of the sides of a spherical triangle are as the sines of their opposite angles. Then CD may be found from BC and C, and AD from AB and A, by means of Napier's theorem. Accordingly AC will be known, being the sum or difference of AD and CD. Lastly ABC may be determined, by means of AC with another side and the opposite angle, from the above mentioned theorem concerning the sides and opposite angles of a triangle.

The following are the formulæ according to which the computations are to be made.

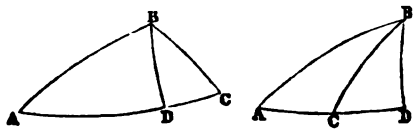
$$\sin BC\ (= \sin AB \sin A \div \sin C)$$
$$= \sin AB \sin A \operatorname{cosec} C$$
$$\tan AD = \cos A \tan AB$$
$$\tan CD = \cos C \tan BC$$
$$\sin ABC\ (= \sin A \sin AC \div \sin BC)$$
$$= \sin A \sin AC \operatorname{cosec} BC.$$

Example. Let AB = 57° 25′, A = 11° 12′ 30″, C = 22° 55′; then will BC = 24° 52′ 24″, AD = 56° 54′ 50″, CD = 23° 7′ 26″, AC = 80° 2′ 16″, ABC = 152° 55′ 30″.

20. When the three sides of a triangle are given, and the angles required, there is occasion to use the following theorem.

If a perpendicular be drawn from either vertex of a triangle to the opposite side or base, the tangent of half the sum of the other two sides into the tangent of half their difference, will be equal to the tangent of half the sum of the segments of the base into the tangent of half their difference.

This proposition may be demonstrated thus.



In the spherical triangle ABC draw BD opposite A and perpendicular to AC: let M stand for half of (AB+BC), and N for half of (AB−BC); also let M′ be half of (AD+DC), and N′ half of (AD−DC). Then it is to be proved that $\tan M \tan N = \tan M' \tan N'$.

In page 72 it is shown that

$$\cos BC \cos AD = \cos AB \cos CD.$$

But obviously,

$$AB = M+N,\ BC = M-N,\ AD = M'+N',\ \text{and}\ CD = M'-N':$$

therefore

$$\cos(M-N)\cos(M'+N') = \cos(M+N)\cos(M'-N'),$$

or
$$\frac{\cos(M-N)}{\cos(M+N)} = \frac{\cos(M'-N')}{\cos(M'+N')};$$

that is, (Day's Trigon. Art. 208),

$$\frac{\cos M \cos N + \sin M \sin N}{\cos M \cos N - \sin M \sin N} = \frac{\cos M' \cos N' + \sin M' \sin N'}{\cos M' \cos N' - \sin M' \sin N'}:$$

then if the numerator and denominator of the first fraction be divided by cos M cos N, and those of the second by cos M′ cos N′, and tangents be put for sines divided by cosines, there will result the equation,

$$\frac{1+\tan M \tan N}{1-\tan M \tan N}=\frac{1+\tan M' \tan N'}{1-\tan M' \tan N'},$$

which reduced gives the equation that was to be determined, namely,

$$\tan M \tan N=\tan M' \tan N'.$$

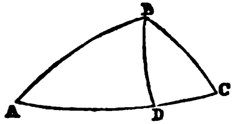
The case in which the sides of a triangle are given and the angles required, and that in which the angles are given and the sides required, are now to be considered: and they will be treated of in the order in which they are here mentioned.

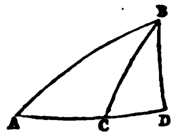
21. *Given the three sides of a spherical triangle; to find the three angles.*

Let ABC be a spherical triangle whose sides are given; it is required to find the angles A, B, C. From B draw BD opposite to A, and perpendicular to AC or AC produced: then by the preceding theorem,

$$\tan \tfrac{1}{2}(AB+BC) \tan \tfrac{1}{2}(AB-BC)=\tan \tfrac{1}{2}(AD+DC) \tan \tfrac{1}{2}(AD-DC);$$

from which equation may be found the segments AD and DC: for if D is in AC, the sum of the segments is known, being equal to AC; and as the first side of the equation consists of known quantities, the value of $\tan \frac{1}{2}(AD-DC)$, and thence also that of $\frac{1}{2}(AD-DC)$ may be determined; and if D is in AC produced, the difference of the segments is equal to AC, and then the value of half the sum of the segments may be found. Half the sum increased by half the difference will give the greater segment, and diminished by half the difference, it will give the less. Then by Napier's rule, the angle A may be found from the side AB and the segment AD, and the angle C from the side BC and the segment CD. And lastly, the angle ABC may be obtained by means of the theorem in Art. 15.





The formulæ to be used in the present case are then as follows:

(1) $\tan x = \tan \frac{1}{2}(AB+BC) \tan \frac{1}{2}(AB-BC) \div \tan \frac{1}{2} AC$; where x is an arc to be computed; and it must be so taken as to be greater or less than a quadrant, according as $\frac{1}{2}(AB+BC)$ is greater or less than a quadrant, AB being supposed greater than BC.

(2) $$\cos A = \cot AB \tan AD,$$
where $AD = x + \frac{1}{2}AC$, 180° being rejected, if the sum happen to exceed this, as it may when $\frac{1}{2}(AB+BC)$ is greater than a quadrant.

(3) $$\cos BCD = \cot BC \tan CD;$$
where $$CD = AC \sim AD.$$

(4) $$\sin ABC \; (= \sin A \sin AC \div \sin BC) = \sin A \sin AC \operatorname{cosec} BC.$$

Example. Let AB=120° 50′, BC=85° 40′, AC=37° 30′; then x=104° 9′ 23″ (x being greater than 90° because $\frac{1}{2}(AB+BC)$ is so), AD=122° 54′ 23″, A=22° 42′ 45″, (A being less than 90° because its cosine is positive, inasmuch as the two factors of which it is the product, namely cot AB and tan AD, are both negative), CD=85° 24′ 23″, BCD=19° 25′ 6″, ACB=160° 34′ 54″, ABC=13° 38′ 3″.

22. *Given the three angles of a spherical triangle; to find the sides.*

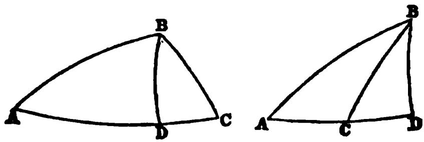
The supplements of the given angles are the sides of a triangle polar to the one in question. The sides of this polar triangle being accordingly known, its angles may be computed by the method just given: and then the supplements of these angles may be taken for the required sides of the other triangle, expressed in degrees, minutes, and seconds.

Example. In a spherical triangle ABC, let A=149° 55′, B=103° 15′, C=131° 6′; then if A′B′C′ be the polar triangle, B′C′=30° 5′, A′C′=76° 45′, A′B′=48° 54′; whence x'=9° 47′ 2″, A′D′=48° 9′ 32″, A′=13° 1′ 32″, C′D′=28° 35′ 28″, B′C′D′=A′C′B′=19° 48′ 29″, A′B′C′=154° 2′ 43″; and therefore BC (=180°−A′)=166° 58′ 28″, AC=25° 57′ 17″, AB=160° 11′ 31″. The angle A′B′C′ must be obtuse to make the sum of the angles in the triangle A′B′C′ exceed 180°.

23. In the six preceding articles are furnished methods of calculation for all the cases of oblique-angled trigonometry. These methods are for the most part sufficiently simple, and such as may be readily called to mind. But in one or two particulars they may perhaps be more clearly understood and more easily remembered from the following general statement.

Let the two right-angled triangles into which an oblique-angled triangle is divided by a perpendicular from one of its vertices to the opposite side, be called *contiguous* right-angled triangles; and designate as *corresponding parts* in these triangles, either the segments of the base, the segments of the vertical angle, the two hypotenuses, or the two angles opposite the perpendicular.

Thus if BD be perpendicular to AC in the triangle ABC, ABD and BDC will be *contiguous* triangles; and AD and DC will be *corresponding* parts in these triangles, as also ABD and CBD, AB and BC, or A and C.



Now if two parts of one triangle and one of the corresponding parts in the contiguous triangle be known, the remaining part may be found by means of Napier's theorem, as follows. The perpendicular, which is common to the two triangles, may be expressed in terms of the two parts given in the first triangle, and likewise in terms of the corresponding parts in the other triangle; and then an equation may be formed of the two expressions, which will involve only the two given parts of one triangle, and the corresponding parts of the other: and as one of these last is known, the remaining part, which is unknown, may be found from the equation.

The cases to be considered here are six in number: for the given parts in one of the contiguous triangles, as ABD, may be

(1) AB and AD
(2) AB and A
(3) AB and ABD
(4) AD and A
(5) AD and ABD
(6) A and ABD.

CASE 1. *Given* AB *and* AD *in the triangle* ABD, *and one of the corresponding parts* CB *and* CD *in the contiguous triangle* BCD; *to find the remaining part.*

If BD be taken in connection with AB and AD, the equation involving these parts, is according to Napier's theorem as follows:

$$\cos AB = \cos AD \cos BD;$$

whence $$\cos BD = \frac{\cos AB}{\cos AD}.$$

In like manner it is found that

$$\cos BD = \frac{\cos BC}{\cos CD}.$$

Hence $$\frac{\cos BC}{\cos CD} = \frac{\cos AB}{\cos AD};$$

and from this equation, if AB and AD are known, with either of the corresponding parts BC and CD, the remaining part may be found: thus if CD be known,

$$\cos BC = \cos CD \times \frac{\cos AB}{\cos AD},$$

and if BC is known,

$$\cos CD = \cos BC \times \frac{\cos AD}{\cos AB}.$$

CASE 2. *Given* AB *and* A, *and one of the corresponding parts* BC *and* C; *to find the other.*

By Napier's theorem, the equation connecting the parts AB, A and BD, is this:

$$\sin BD = \sin AB \sin A.$$

Also $$\sin BD = \sin BC \sin C:$$

whence, $\sin BC \sin C = \sin AB \sin A$,

or if BC be the unknown quantity,

$$\sin BC = \sin AB \times \frac{\sin A}{\sin C};$$

and if C be unknown,

$$\sin C = \sin A \times \frac{\sin AB}{\sin BC}.$$

The equation $\sin BC \sin C = \sin AB \sin A$, converted into a proportion, gives the theorem of Art. 15.

CASE 3. *Given* AB *and* ABD *in the triangle* ABD, *and one of the corresponding parts* BC *and* CBD; *to find the other part.*

The equation involving the quantities AB, ABD and BD, is by Napier's theorem the following,

$$\cos ABD = \text{cotan } AB \tan BD;$$

whence, $\tan BD \ (= \cos ABD \div \text{cotan } AB)$

$$= \cos ABD \tan AB.$$

In like manner it will appear that

$$\tan BD = \cos CBD \tan BC.$$

Hence, $\cos CBD \tan BC = \cos ABD \tan AB.$

If then ABD, AB and BC are known, CBD may be found from the equation,

$$\cos CBD = \cos ABD \times \frac{\tan AB}{\tan BC};$$

and if ABD, AB and CBD are known, BC may be found from this equation,

$$\tan BC = \tan AB \times \frac{\cos ABD}{\cos CBD}.$$

CASE 4. *Given* AD *and* A, *with one of the corresponding parts* CD *and* C; *to find the other.*

The equations for this case are as follows,

$$\sin AD = \tan BD \cot A$$

$$\tan BD \ (= \sin AD \div \cot A)$$

$$= \sin AD \tan A;$$

also $\tan BD = \sin CD \tan C;$

and therefore $\sin CD \tan C = \sin AD \tan A;$

or if CD be unknown,

$$\sin CD = \sin AD \times \frac{\tan A}{\tan C},$$

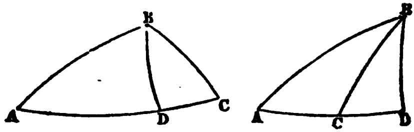
and if C be unknown,

$$\tan C = \tan A \times \frac{\sin AD}{\sin CD}.$$

CASE 5. *Given* AD *and* ABD, *with either of the parts* CD *and* CBD; *to find the other of these parts.*

The equations for this case arranged as in the preceding cases are

$$\sin BD = \tan AD \cot ABD,$$
$$\sin BD = \tan CD \cot CBD,$$
$$\tan CD \cot CBD = \tan AD \cot ABD,$$
$$\tan CD = \tan AD \times \frac{\cot ABD}{\cot CBD},$$
$$\cot CBD = \cot ABD \times \frac{\tan AD}{\tan CD}.$$



The last equation will give the value of CBD when CD is known, and the last but one will give the value of CD when CBD is known.

From the equation tan CD cot CBD = tan AD cot ABD, may be derived a proportion that is worthy of notice. Since the cotangent is the reciprocal of the tangent of an angle,

$$\frac{\tan CD}{\tan CBD} = \frac{\tan AD}{\tan ABD},$$

whence, $\tan AD : \tan CD :: \tan ABD : \tan CBD$;
which may in other terms be stated thus,

If a perpendicular be drawn from the vertex of a spherical triangle to the opposite side or base, the tangents of the segments of the base are as the tangents of the segments of the vertical angle.

CASE 6. *Given* A *and* ABD, *with one of the parts* C *and* CBD; *to find the other.*

Here the equations concerned are,

$$\cos A = \cos BD \sin ABD,$$

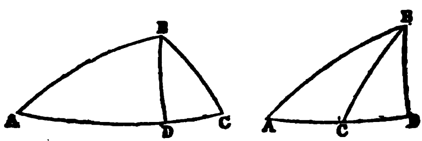
or, $$\cos BD = \cos A \div \sin ABD,$$

also, $$\cos BD = \cos C \div \sin CBD,$$

and therefore, $$\frac{\cos C}{\sin CBD} = \frac{\cos A}{\sin ABD};$$

or finally, $$\cos C = \cos A \times \frac{\sin CBD}{\sin ABD},$$

and $$\sin CBD = \sin ABD \times \frac{\cos C}{\cos A}.$$



It is not important that the final equations in these six cases be remembered, if the general method of obtaining them be kept in mind.

24. For all these cases, except the second and fifth, Dr. Bowditch has furnished one or two rules of a very simple form and of easy application. They are similar to Napier's rules of the circular parts in a right-angled triangle; from which they are deduced directly in the following manner.

In the second and fifth of the preceding cases, the common perpendicular BD is the *middle part;* but in each of the other four cases, some other part is the middle one. Now if in the triangle ABD one of the other four circular parts be taken for a middle part, and the corresponding part in BDC be taken for the middle one, BD will be an adjacent part in each triangle or an opposite part in each. And the rules in question are, that the sines of the middle parts will be, in the former case, as the tangents of the other two adjacent parts, and in the latter case, as the cosines of the other two opposite parts; or to state them concisely,

In two contiguous right-angled triangles, the sines of the middle parts are as the tangents of the adjacent parts, or as the cosines of the opposite parts.

By Napier's theorem the sine of the middle part in each triangle equals the product of the tangents of the adjacent parts, and from the two equations thus furnished may be formed the proportion, that the sine of the middle part in one triangle is to the sine of the middle part in the other, as the product of the tangents of the adjacent parts in the former triangle to the product of the tangents of the adjacent parts in the latter. And if the tangent of BD be one of the factors in each product, we may reject it without destroying the proportion; and hence it appears that the sines of the middle parts are as the tangents of the adjacent parts: which is one of the rules that was to be deduced.

In a similar way it is proved that the sines of the middle parts are as the cosines of the opposite parts.*

25. The method of applying these rules is as follows.

(1) If comp. AB be taken for the middle part in the triangle ABD and the corresponding part comp. BC for the middle one in BCD, then BD is an opposite part in each triangle, and the other opposite parts are AD and DC:

hence, sin comp. AB : sin comp. BC :: cos AD : cos CD,

or cos AB : cos BC :: cos AD : cos CD.

This proportion answers to the final equation in Case 1.

(2) If comp. A and comp. C be the middle parts, BD is an opposite part, and the other parts are comp. ABD, comp. CBD.

Hence cos A : cos C :: cos comp. ABD : cos comp. CBD

or cos A : cos C :: sin ABD : sin CBD,

a proportion which answers to the last equation in Case 6.

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* Bowditch's Rules may be demonstrated more concisely thus.

Let M stand for a middle part in the triangle ABD, N for an adjacent and O for an opposite part; and let the corresponding parts in the contiguous triangle be M′, N′, O′; also let P stand for the common perpendicular BD.

Then if P be an adjacent part, the equations involving P, are by Napier's theorem, sin M = tan N tan P, sin M′ = tan N′ tan P;

whence, $$\frac{\sin M}{\sin M'} = \frac{\tan N}{\tan N'},$$

or, sin M : sin M′ : : tan N : tan N′; as was to be proved.

And if P be an opposite part, the equations involving P are sin M = cos O cos P, sin M′ = cos O′ cos P;

whence, $$\frac{\sin M}{\sin M'} = \frac{\cos O}{\cos O'},$$

or, sin M : sin M′ : : cos O : cos O′; as was also to be proved.

(3) If AD and DC are the middle parts, BD is an adjacent part, and the other adjacent parts are comp. A, comp. C:

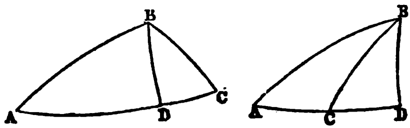
then sin AD : sin CD :: cot A : cot BCD,

which answers to the final equation in Case 4.

(4) If comp. ABD and comp. CBD be the middle parts, BD is an adjacent part, and the other adjacent parts are comp. AB, comp. BC:

hence, cos ABD : cos CBD :: cot AB : cot BC,

which answers to the final equation in Case 3.



In each of the above proportions there are four quantities concerned, two belonging to each triangle. Now if it is proposed to find the proportion involving these quantities, *and the middle parts be not specified*, they must be so chosen, that the other two parts shall be of the same name with the perpendicular BD, whether adjacent or opposite. Thus if the quantities involved are AB, AD, BC, CD; comp. AB and not AD must be one of the middle parts, and comp. BC must be the other; for if AD were taken for the middle part, BD would be an adjacent and comp. AB an opposite part.

26. The preceding methods of solution may be made to answer for all cases that can occur in Spherical Trigonometry. There are certain formulæ, however, which have not been given, that are often useful in trigonometrical calculations. These may be most conveniently arrived at by algebraic processes. Indeed all the formulæ relating to the sides and angles of spherical triangles may thus be easily derived from a single theorem. In the following articles it is designed to deduce the most important of them in this way; it is intended also to point out the best methods of resolving triangles for the different cases in Spherical Trigonometry; to notice the limitations to which the values of sides and angles of a triangle may be subject, and to give rules which will, whenever the case allows, preclude the uncertainty whether a side or angle sought is greater or less than 90°.

It will be convenient to have here certain formulæ belonging to plane trigonometry, which are often referred to. They are taken, as the accompanying references show, from Day's Mathematics.

Let a be any arc or angle, and let Radius be unity: then

(1) $$\sin^2 a+\cos^2 a=1$$

(2) $$\sec^2 a-\tan^2 a=1$$

(3) $$\text{cosec}^2 a-\cot^2 a=1$$

(4) $$\frac{\sin a}{\cos a}=\tan a$$

(5) $$\frac{\cos a}{\sin a}=\cot a$$

(6) $$\tan a \cot a=1$$

(7) $$\sin a \,\text{cosec}\, a=1$$

(8) $$\cos a \sec a=1.$$

Again, let a and b be any two arcs or angles: then

(9) $$\sin(a+b)=\sin a\cos b+\sin b\cos a$$

(10) $$\sin(a-b)=\sin a\cos b-\sin b\cos a$$

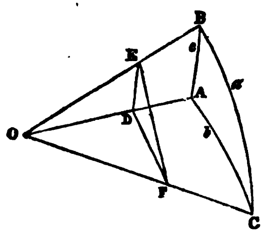
(11) $$\cos(a+b)=\cos a\cos b-\sin a\sin b$$

(12) $$\cos(a-b)=\cos a\cos b+\sin a\sin b.$$

(Day's Trigon. Arts. 93. 94. 208).

A theorem is now to be investigated from which the various formulæ of spherical trigonometry can be conveniently deduced.

27. Let ABC be a triangle on the surface of a sphere whose center is O; draw the radii OA, OB, OC; from D any point in the line OA draw DE and DF at right angles to that line, in the planes ABO, ACO; then is EDF the angle of inclination between these planes, and therefore equal to the spherical angle BAC.



Now in the plane triangle EDF,

$$EF^2 = ED^2 + DF^2 - 2ED \times DF \cos EDF,$$

(Day's Trigon. Art. 234, Theor. B.),

and in the triangle EOF,

$$EF^2 = EO^2 + OF^2 - 2EO \times OF \cos EOF:$$

hence $0 =$

$$\left. \begin{array}{r} EO^2 - ED^2 + OF^2 - DF^2 - 2EO \times OF \cos EOF \\ + 2ED \times DF \cos EDF. \end{array} \right\}$$

But (Euc. 47, 1,) $EO^2 - ED^2 = OD^2$, and $OF^2 - DF^2 = OD^2$: by reduction therefore

$$EO \times OF \cos EOF = OD^2 + ED \times DF \cos EDF,$$

or $$\cos EOF = \frac{OD^2}{EO \times OF} + \frac{ED}{EO} \times \frac{DF}{OF} \cos EDF.$$

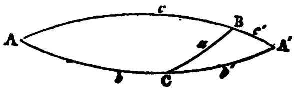
Now $\dfrac{OD^2}{EO \times OF}$ equals $\dfrac{OD}{EO} \times \dfrac{OD}{OF}$, which equals $\cos EOD \times \cos FOD$ (Day's Trigon.):

also $\dfrac{ED}{EO} \times \dfrac{DF}{OF}$ equals $\sin EOD \times \sin FOD$.

But the angles EOF, FOD, EOD, are measured by the arcs subtending them, namely a, b, c, the sides of the triangle ABC: then by substitution

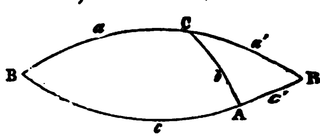
$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

It will be seen by referring to the figure here used, that we have proceeded on the supposition that the angles AOB, AOC, subtended by the arcs c, b, are acute; but the formula which was obtained will hold good for other cases, as may be shown in the following manner.



First let ABC be a spherical triangle having each of its sides b and c greater than a quadrant, let these sides produced meet in A'; then will a second triangle A'BC be formed having one side a in common with the first, having its angle A' equal to A, and the two sides containing this angle each less than a quadrant, since they are the supplements of b and c. Then if these sides be represented by b', c', we have the equation $\cos a = \cos b' \cos c' + \sin b' \sin c' \cos A'$.

Now $\cos b' = -\cos b$, and $\cos c' = -\cos c$, since the cosines of an angle and of its supplement are equal and of opposite signs: therefore $\cos b' \cos c' = \cos b \cos c$. And since the sines of angles are the same as the sines of their supplements, $\sin b' \sin c' = \sin b \sin c$. Hence $\cos a = \cos b \cos c + \sin b \sin c \cos A$; as was to be shown.



Again, let ABC be a spherical triangle in which the side b is less and the side c greater than a quadrant. Let the sides a and c produced meet in B'; thus will be formed a second triangle AB'C having an angle

B'AC which is the supplement of BAC, and having one of its sides b in common with the triangle ABC, its other sides a', c', being the supplements of a and c: c' is therefore less than a quadrant, and b being also less than a quadrant,

$$\cos a' = \cos b \cos c' + \sin b \sin c' \cos B'AC.$$

But $\cos a' = -\cos a$, $\cos c' = -\cos c$, $\sin c' = \sin c$,

and $\cos B'AC = -\cos BAC$,

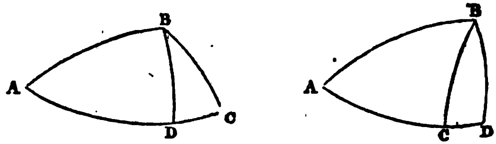
$$\therefore -\cos a = -\cos b \cos c - \sin b \sin c \cos BAC,$$

or $\cos a = \cos b \cos c + \sin b \sin c \cos A$; which is the same formula as was first obtained.

We have thus far supposed each of the sides b and c to be either greater or less than a quadrant.

It remains to consider the case in which one of them is a quadrant, and also that in which they are both quadrants.

It is obvious that the formula in question will hold good in the latter case; for then $\cos b$ and $\cos c$ are each equal to zero, the sines of b and c are each equal to unity; and $\cos a = \cos A$, since a is evidently the measure of the angle A.



But if one of the sides about the angle A, namely AB be equal, and the other AC unequal to a quadrant, from AC or AC produced cut off AD equal to a quadrant and draw the arc BD. This will be the measure of the angle A, and the angle BDC will be a right angle. Then, if BD be not a quadrant, we shall have, as in preceding cases,

$$\cos BC = \cos BD \times \cos CD + \sin BD \times \sin CD \cos D.$$

But cos D equals nothing; and cos CD, that is, cos (AC ∽ AD) equals cos AC cos AD + sin AC sin AD (Art. 26, Equat. 12), which equals sin AC, since AD is a quadrant. If then we substitute A for BD its measure, a for BC, and b for AC, we get the equation

$$\cos a = \cos A \sin b.$$

This equation is obviously that which will be furnished by the general formula first obtained, upon the supposition that c is a quadrant; for then $\cos c = 0$, and $\sin c = 1$.

In the only remaining case, when AB and BD are both quadrants, B is the pole of AC, BC or a is a quadrant and A is a right angle; in which case $\cos a$, $\cos c$, and cos A are each equal to nothing.

Hence the formula $\cos a = \cos b \cos c + \sin b \sin c \cos A$, is applicable to all kinds of spherical triangles. In the form of a theorem it may be stated thus:

The cosine of one of the sides of a spherical triangle, is equal to the product of the cosines of the other two sides, increased by the product of their sines multiplied into the cosine of the included angle.

There are three equations answering to this theorem, for every triangle: thus,

(1) $\cos a = \cos b \cos c + \sin b \sin c \cos A$

(2) $\cos b = \cos a \cos c + \sin a \sin c \cos B$

(3) $\cos c = \cos a \cos b + \sin a \sin b \cos C.$

28. In each trigonometrical calculation, there are four parts concerned, three given and one required;

and as the three equations at the end of the last article together contain all the six parts of a triangle, we may obviously deduce from these equations an equation involving any four of these parts, by eliminating the other two. If for instance we wish to obtain an equation involving the three angles A, B, C, and the side b; we may combine the first equation with the second, to obtain an equation free from a, and then combine the first and third equations, to form another equation free from a: having thus found two equations from which a is excluded, we may finally combine these to form an equation from which c also shall be eliminated; and this will be the equation required. Now the only cases that here require to be distinguished, are the following; namely, those in which the four parts concerned are

(1) The three sides and one angle

(2) Two sides and the opposite angles

(3) Two sides, the included angle and one of the opposite angles

(4) The three angles and one side.

There are six equations belonging to the third case and three to each of the others.

The formulæ for the first case have been given in the last article; and from them the formulæ for the other cases are now to be deduced.

29. To find an equation involving two sides a, b, and their opposite angles A, B, it is sufficient to combine the first two equations at the end of Art. 27, so as to eliminate c, since C is not found in either of them. This might be done by first forming a new equation containing sin c, but free from cos c; and then another equation containing cos c but not sin c. From the former equation the value of sin c, and from the latter that of cos c would be known: and the squares of these values made equal to unity, would give the equation required. But there is another method of procedure, which though less direct and obvious, is more simple, and has the advantage of bringing to view formulæ that are useful for other purposes. It is as follows:

From equation (1) Art. 27,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

hence
$$1 + \cos A = \frac{\cos a - \cos b \cos c + \sin b \sin c}{\sin b \sin c}$$

$$= \frac{\cos a - \cos(b+c)}{\sin b \sin c} \quad \text{(Art. 26, Equat. 11).}$$

If for convenience of reduction, we substitute

$x-y$ for a, and $x+y$ for $b+c$, then

$$\cos a - \cos(b+c) = \cos(x-y) - \cos(x+y)$$

$$= 2 \sin x \sin y \quad \text{(Arts. 26, Equats. 11, 12).}$$

If we now substitute for x and y their values in terms of a, b, and c,

$$1 + \cos A = \frac{2 \sin \frac{1}{2}(a+b+c) \sin \frac{1}{2}(b+c-a)}{\sin b \sin c};$$

and if we put h for half the sum of the sides a, b and c,

$$(1) \qquad 1 + \cos A = \frac{2 \sin h \sin(h-a)}{\sin b \sin c}.$$

Again,

$$1 - \cos A = \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos(b-c) - \cos a}{\sin b \sin c};$$

and by a method similar to the preceding, we find

$$1-\cos A=\frac{2\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a+c-b)}{\sin b\sin c},$$

or

$$(2)\quad 1-\cos A=\frac{2\sin(h-b)\sin(h-c)}{\sin b\sin c}.$$

If we multiply the corresponding members of equations (1) and (2), substitute $\sin^2 A$ for $1-\cos^2 A$, and extract the square root, we have the equation

$$(3)\quad \sin A=\frac{2\sqrt{\sin h\sin(h-a)\sin(h-b)\sin(h-c)}}{\sin b\sin c};$$

and if for the sake of symmetry we divide each side by $\sin a$, we shall obtain

$$(4)\quad \frac{\sin A}{\sin a}=\frac{2\sqrt{\sin h\sin(h-a)\sin(h-b)\sin(h-c)}}{\sin a\sin b\sin c}.$$

As the second member involves all the three sides of the triangle in the same way, it must be equal to $\frac{\sin B}{\sin b}$ and to $\frac{\sin C}{\sin c}$, as well as to $\frac{\sin A}{\sin a}$. Hence,

$$(5)\quad \frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}.$$

We have here equations each involving two sides of a triangle and the opposite angles; as was required. The theorem answering to them is this:

The sines of the sides of a spherical triangle are as the sines of the opposite angles.

It has been otherwise proved in Art. 15.

30. To find next an equation involving two sides a and b, the included angle C, and one of the opposite angles B: first substitute for $\cos c$ in the second equation at the end of Art. 27, its value as given by the third equation; the result will be

$\cos b = \cos^2 a \cos b + \cos a \sin a \sin b \cos C + \sin a \sin c \cos B$;
and if we transpose $\cos^2 a \cos b$, and substitute $\sin^2 a$ for $1-\cos^2 a$, the first member of the equation will be $\cos b \sin^2 a$; and if each member be divided by $\sin a$, then

$$\cos b \sin a = \cos a \sin b \cos C + \sin c \cos B.$$

But by equation (5) of the last article, $\sin c = \sin b \frac{\sin C}{\sin B}$: substituting this value, dividing by $\sin b$, and putting cotangents for cosines divided by sines, we obtain

$$(1)\quad \cot b \sin a = \cos a \cos C + \sin C \cot B.$$

In like manner may be obtained the following equations:

$$(2)\quad \cot a \sin b = \cos b \cos C + \sin C \cot A,$$
$$(3)\quad \cot c \sin b = \cos b \cos A + \sin A \cot C,$$
$$(4)\quad \cot b \sin c = \cos c \cos A + \sin A \cot B,$$
$$(5)\quad \cot a \sin c = \cos c \cos B + \sin B \cot A,$$
$$(6)\quad \cot c \sin a = \cos a \cos B + \sin B \cot C;$$

which are all the equations that taken singly involve two sides, the included angle and one of the opposite angles in a triangle.

31. It remains to find an equation involving the three angles and one of the sides of a triangle. If with the equation

$$\cos b \sin a = \cos a \sin b \cos C + \sin c \cos B$$

found in the preceding article, we combine the analogous one,

$$\cos a \sin b = \cos b \sin a \cos C + \sin c \cos A,$$

(obtained from the former by interchanging the sides a and b, and the angles A and B,) and eliminate $\sin b$, the result is

$$\cos b \sin a = \cos b \sin a \cos^2 C + \sin c \cos A \cos C + \sin c \cos B.$$

And if we transpose $\cos b \sin^2 a \cos^2 C$, substitute $\sin^2 C$ for $1-\cos^2 C$, divide by $\sin c$, and substitute for $\frac{\sin a}{\sin c} \sin C$ its value $\sin A$ obtained from equation (4) of the last article, we shall find that

$$\cos b \sin A \sin C = \cos A \cos C + \cos B,$$

and by transposition,

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b:$$

which shows that

The cosine of one of the angles of a spherical triangle is equal to the product (taken negatively) of the cosines of the other two angles added to the product of their sines multiplied into the cosine of the intermediate side.

There are three equations comprised in this theorem; namely,

(1) $\cos A = -\cos B \cos C + \sin B \sin C \cos a$

(2) $\cos B = -\cos A \cos C + \sin A \sin C \cos b$

(3) $\cos C = -\cos A \cos B + \sin A \sin B \cos c$.

32. The equations which have now been obtained will give the value of either of the six parts of a triangle in terms of the other three; for which purpose fifteen equations are evidently requisite: but all of them except the equations numbered (5) in Art. 29 are ill suited for logarithmic calculations. We must therefore seek for other formulæ more convenient.

By Art. 210, Day's Trigon.

$$1+\cos A = 2\cos^2 \tfrac{1}{2}A, \qquad 1-\cos A = 2\sin^2 \tfrac{1}{2}A.$$

Then by substitution in equations (1) and (2) of Art. 29, and by reduction, we obtain the following results.

(1) $\cos \frac{1}{2}A = \sqrt{\frac{\sin h \sin (h-a)}{\sin b \sin c}}$

(2) $\sin \frac{1}{2}A = \sqrt{\frac{\sin (h-b) \sin (h-c)}{\sin b \sin c}}$;

whence also by division and putting $\tan \frac{1}{2}A$ for $\frac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A}$,

(3) $\tan \frac{1}{2}A = \sqrt{\frac{\sin (h-b) \sin (h-c)}{\sin h \sin (h-a)}}$.

The equation (3) of Art. 29 being nearly related to these, we will place it here with them.

(4) $\sin A = \frac{2\sqrt{\sin h \sin (h-a) \sin (h-b) \sin (h-c)}}{\sin b \sin c}$.

These four equations are all convenient for logarithmic calculations.

33. The equations of the last article express the value of an angle of a triangle in terms of the sides. There are corresponding equations by which a side of a triangle is expressed in terms of the angles. These may be deduced from the equations at the end of Art. 31, in nearly the same way as the preceding equations have been derived from those of Art. 27. It will be sufficient here to state them without the investigation. They are the following.

(1) $\cos \frac{1}{2} a = \sqrt{\frac{\cos (H-B) \cos (H-C)}{\sin B \sin C}}$

(2) $\sin \frac{1}{2} a = \sqrt{\frac{-\cos H \cos (H-A)}{\sin B \sin C}}$

(3) $\tan \frac{1}{2} a = \sqrt{\frac{-\cos H \cos (H-A)}{\cos (H-B) \cos (H-C)}}$

(4) $\sin a = \frac{2\sqrt{-\cos H \cos (H-A) \cos (H-B) \cos (H-C)}}{\sin B \sin C}$,

where $$H=\tfrac{1}{2}(A+B+C);$$
and if H′ be put for H − 90°, then

$$(5)\qquad \cos\tfrac{1}{2}a=\sqrt{\frac{\sin(B-H')\sin(C-H')}{\sin B\sin C}}$$

$$(6)\qquad \sin\tfrac{1}{2}a=\sqrt{\frac{\sin H'\sin(A-H')}{\sin B\sin C}}$$

$$(7)\qquad \tan\tfrac{1}{2}a=\sqrt{\frac{\sin H'\sin(A-H')}{\sin(B-H')\sin(C-H')}}$$

$$(8)\ \sin a=\frac{2\sqrt{\sin H'\sin(A-H')\sin(B-H')\sin(C-H')}}{\sin B\sin C};$$

which last are a little more convenient for use than the first four equations.

34. When from the vertex B of a spherical triangle ABC, a perpendicular BD is drawn opposite A, to the side AC or AC produced, it has been shown in Art. 20 that

$$\tan\tfrac{1}{2}(AD+DC)\tan\tfrac{1}{2}(AD-DC)=\tan\tfrac{1}{2}(AB+BC)\tan\tfrac{1}{2}(AB-BC),$$

or as it may be written,

$$(1)\ \tan(AD-\tfrac{1}{2}AC)=\tan\tfrac{1}{2}(AB+BC)\tan\tfrac{1}{2}(AB-BC)\div\tan\tfrac{1}{2}AC.$$

In a similar way it might be shown that

$$(2)\ \tan(\tfrac{1}{2}ABC-ABD)=\tan\tfrac{1}{2}(A+C)\tan\tfrac{1}{2}(A-C)\times\tan\tfrac{1}{2}ABC.$$

It may also be shown that

$$\tan\tfrac{1}{2}(ABD+CBD)\tan\tfrac{1}{2}(ABD-CBD)=\frac{\sin(AB-BC)}{\sin(AB+BC)},$$

or

$$(3)\qquad \tan(ABD-\tfrac{1}{2}ABC)=\frac{\sin(AB-BC)}{\sin(AB+BC)}\cot\tfrac{1}{2}ABC;$$

and that

$$(4)\qquad \tan(\tfrac{1}{2}AC-AD)=\frac{\sin(A-C)}{\sin(A+C)}\tan\tfrac{1}{2}AC;$$

which equations are suitable for logarithmic calculations.

35. From the equations in Art. 32 numbered (1) and (2), with the corresponding equations for the angles B and C instead of A, we readily derive the following:

$$(1)\qquad \frac{\cos\frac{1}{2}A\cos\frac{1}{2}B}{\sin\frac{1}{2}C}=\frac{\sin h}{\sin c}$$

$$(2)\qquad \frac{\sin\frac{1}{2}A\sin\frac{1}{2}B}{\sin\frac{1}{2}C}=\frac{\sin(h-c)}{\sin c}$$

$$(3)\qquad \frac{\sin\frac{1}{2}A\cos\frac{1}{2}B}{\cos\frac{1}{2}C}=\frac{\sin(h-b)}{\sin c}$$

$$(4)\qquad \frac{\cos\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}C}=\frac{\sin(h-a)}{\sin c}.$$

Combining the first two of these by addition, we obtain

$$\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}=\frac{\sin h+\sin(h-c)}{\sin c},$$

or if we restore the value of h, develop and reduce, using $2\sin\frac{1}{2}c\cos\frac{1}{2}c$ instead of $\sin c$,

$$(5)\qquad \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}.$$

From the same two equations, by subtraction instead of addition, we derive the following,

$$(6)\qquad \frac{\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}C}=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}.$$

And in a similar way from equations (3) and (4) we obtain

$$(7)\qquad \frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}C}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}.$$

$$(8)\qquad \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}.$$

36. From the equations in Art. 33, numbered (5) and (6), with the corresponding ones for b and c instead of a, we may derive other equations resembling those of the last article; thus,

$$(1)\qquad \frac{\sin\frac{1}{2}a\sin\frac{1}{2}b}{\cos\frac{1}{2}c}=\frac{\sin H'}{\sin C}=\frac{-\cos H}{\sin C}$$

$$(2)\qquad \frac{\cos\frac{1}{2}a\cos\frac{1}{2}b}{\cos\frac{1}{2}c}=\frac{\sin(C-H')}{\sin C}=\frac{\cos(H-C)}{\sin C}$$

$$(3)\qquad \frac{\cos\frac{1}{2}a\sin\frac{1}{2}b}{\sin\frac{1}{2}c}=\frac{\sin(B-H')}{\sin C}=\frac{\cos(H-B)}{\sin C}$$

$$(4)\qquad \frac{\sin\frac{1}{2}a\cos\frac{1}{2}b}{\sin\frac{1}{2}c}=\frac{\sin(A-H')}{\sin C}=\frac{\cos(H-A)}{\sin C};$$

from which result four equations more, that are identical with the last four of the preceding article.

37. By combining the four equations at the end of Art. 35, we obtain the following:

$$(1)\qquad \tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C$$

$$(2)\qquad \tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C$$

$$(3)\qquad \tan\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c$$

$$(4)\qquad \tan\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c.$$

These equations, reduced to proportions, are called *Napier's analogies.*

They form but three independent equations; and from either the first two or the last two may be derived the following:

$$(5)\qquad \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}(a-b)};$$

that is,

The tangent of half the sum of any two sides of a spherical triangle is to the tangent of half their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.

38. In the following articles it is proposed to present in proper order the formulæ which seem best adapted for the solution of problems in each of the cases of oblique-angled trigonometry, and to give such directions for using them as may be requisite.

It is important, however, to notice first certain general limitations of value to which the parts of a triangle are subject. They are as follows.

(1) Each side and each angle is less than 180°.

(2) The greatest angle exceeds 60°.

(3) The greater angle is opposite the greater side.

(4) The sum of any two angles and the sum of the opposite sides, are each equal to 180°, or each greater or each less than 180°.

(5) $360° > a+b+c > 2a$, a being the longest side.

(6) $180° < A+B+C < 180°+2A$, A being the least angle.

The meaning of the condition (5) is, that the sum of the three sides of a triangle is less than 360°, and that this sum is greater than twice the longest side, or what amounts to the same, that the sum of the two shorter sides is greater than the longest side.

And the meaning of the condition (6) is, that the sum of the three angles is greater than 180°, and that this sum is less than 180° added to twice the least angle, or what amounts to the same, that the sum of the two greater angles is less than 180° added to the least angle, which also amounts to the statement that the difference of any two angles is less than the supplement of the third.

The sixth condition obviously embraces the second.

With these conditions in view, we proceed to give the methods of calculation most appropriate for the different cases of oblique-angled trigonometry.

Case 1.

39. *Given the three sides a, b, c; to find one or more of the angles* A, B, C.

Here the only condition concerning the data besides this, that the sides are each less than 180°, is the following,

$$360° > a+b+c > \text{twice the longest side.}$$

First. If only *one* angle A is sought, it may be found by either of the following formulæ; in which $h=\frac{1}{2}(a+b+c)$.

$$(1) \qquad \cos \tfrac{1}{2}A = \sqrt{\frac{\sin h \sin (h-a)}{\sin b \sin c}}$$

$$(2) \qquad \sin \tfrac{1}{2}A = \sqrt{\frac{\sin (h-b) \sin (h-c)}{\sin b \sin c}}$$

$$(3) \qquad \tan \tfrac{1}{2}A = \sqrt{\frac{\sin (h-b) \sin (h-c)}{\sin h \sin (h-a)}}$$

$$(4) \; \sin A = \frac{2\sqrt{\sin h \sin(h-a) \sin(h-b) \sin(h-c)}}{\sin b \sin c} \; \text{(Art. 32).}$$

In using the last formula, it is to be observed that A is an obtuse angle, when the logarithmic sines of $(h-b)$ and $(h-c)$ together exceed those of h and $(h-a)$. It will sometimes be sufficient, when a is the longest side, to observe that A exceeds 60°, in order to determine whether it is acute or obtuse.

There is no necessary uncertainty as to the value of the angle sought, for the value of half the angle must be less than 90°, and by either of the formulæ (1), (2), (3), this value can be determined.

The angle A may also be found as follows.

From Equat. (1), Art. 27, we have the following:

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \cot b \cot c \left(\frac{\cos a}{\cos b \cos c} - 1 \right)$$

$$= \cot b \cot c (\cos a \sec b \sec c - 1),$$

which may be expressed by means of two other equations, thus

$$(5) \qquad \left\{ \begin{aligned} x &= \cos a \sec b \sec c \\ \cos A &= \cot b \cot c \, (x - 1). \end{aligned} \right.$$

If then we compute x from the first equation, (taking the number answering to the sum of the logarithmic cosine and secants,) and add the logarithm of $(x-1)$ to the cotangents of b and c, the sum will be the logarithmic cosine of A; from which the angle is to be obtained. Attention must be paid to the sign of x, which may be either positive or negative. To determine the sign, we must observe here as elsewhere, that the cosines, secants, tangents and cotangents of arcs or angles greater than 90° are negative.

Secondly. If *two* angles A and B are sought,

(1) We may find A as above, and then find B in a similar way, the letters in the formulæ used, being duly changed,

(2) We may find A as above, and then find B by the equation

$$\sin B=\sin A\,\frac{\sin b}{\sin a}.$$

Care must be taken here to ascertain whether the angle B is acute or obtuse.

(3) The two angles may also be found by the following method, which under a little different form is given in Art. 21.

From equation (2), Art. 27,

$$\cos B=\frac{\cos b-\cos a\cos c}{\sin a\sin c},$$

which by proper changes may be reduced to the form

$$\cos B=\cot a\times\frac{\sin\frac{1}{2}c+\cos\frac{1}{2}c\cot\frac{1}{2}c\tan\frac{1}{2}(a+b)\tan\frac{1}{2}(a-b)}{\cos\frac{1}{2}c-\sin\frac{1}{2}c\cot\frac{1}{2}c\tan\frac{1}{2}(a+b)\tan\frac{1}{2}(a-b)}.$$

Likewise

$$\cos A=\cot b\times\frac{\sin\frac{1}{2}c+\cos\frac{1}{2}c\cot\frac{1}{2}c\tan\frac{1}{2}(b+a)\tan\frac{1}{2}(b-a)}{\cos\frac{1}{2}c-\sin\frac{1}{2}c\cot\frac{1}{2}c\tan\frac{1}{2}(b+a)\tan\frac{1}{2}(b-a)}.$$

And if we substitute for $\cot\frac{1}{2}c\tan\frac{1}{2}(a+b)\tan\frac{1}{2}(a-b)$ an equal tangent denoted by $\tan x$, and accordingly for $\cot\frac{1}{2}c\tan\frac{1}{2}(b+a)\tan\frac{1}{2}(b-a)$ substitute $-\tan x$, then

$$\cos B=\cot a\times\frac{\sin\frac{1}{2}c+\cos\frac{1}{2}c\tan x}{\cos\frac{1}{2}c-\sin\frac{1}{2}c\tan x}$$

$$=\cot a\times\frac{\sin(\frac{1}{2}c+x)}{\cos(\frac{1}{2}c+x)}=\cot a\tan(\tfrac{1}{2}c+x),$$

$$\cos A=\cot b\times\frac{\sin\frac{1}{2}c-\cos\frac{1}{2}c\tan x}{\cos\frac{1}{2}c+\sin\frac{1}{2}c\tan x}$$

$$=\cot b\times\frac{\sin(\frac{1}{2}c-x)}{\cos(\frac{1}{2}c-x)}=\cot b\tan(\tfrac{1}{2}c-x).$$

Hence we have for calculation these formulæ,

$$\begin{cases} \tan x = \cot \tfrac{1}{2}c \tan \tfrac{1}{2}(a+b) \tan \tfrac{1}{2}(a-b) \\ \cos B = \cot a \tan (\tfrac{1}{2}c + x) \\ \cos A = \cot b \tan (\tfrac{1}{2}c - x): \end{cases}$$

and the signs of the quantities must be carefully regarded. (See Day's Trigon. Arts. 192–206.)

Thirdly. If all the three angles are required; after computing A and B by one of the foregoing methods, we may proceed to find C,

(1) By either of the first three formulæ in this article,

(2) By the fourth formula of this article,

(3) By the formula

$$\sin C = \sin B \frac{\sin c}{\sin b} = \sin A \frac{\sin c}{\sin a}.$$

In the last two cases, it is to be observed whether C should be an acute or an obtuse angle.

Case 2.

40. *Given two sides a, b, and the included angle* C; *to find one or more of the remaining parts* c, A, B.

The only condition concerning the data here is that a, b and C be each less than 180°.

First. If only the side c be sought;

(1) It may be found from the equation (3) of Art. 27, namely,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

reduced to the form

$$\cos c = \cos a (\cos b + \sin b \tan a \cos C),$$

or if $\tan x$ be substituted for $\tan a \cos C$;

$$\cos c = \cos a\,(\cos b + \sin b \tan x)$$
$$= \frac{\cos a \cos (b \sim x)}{\cos x};$$

the formulæ for computation being accordingly

$$\begin{cases} \tan x = \tan a \cos C \\ \cos c = \dfrac{\cos a \cos (b \sim x)}{\cos x}. \end{cases}$$

(2) The side c may also be found from the equation

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

reduced to these two,

$$\begin{cases} x = \tan a \tan b \cos C \\ \cos c = \cos a \cos b\,(1 + x). \end{cases}$$

Secondly. If one of the unknown angles A only is sought; it may be found by means of the equation (2) in Art. 30; namely,

$$\cot a \sin b = \cos b \cos C + \sin C \cot A,$$

an equation involving a, b, C, A, the four quantities to be concerned in the computation. But this must be reduced to another form, to be suitable for use in logarithmic calculations.

If we transpose and divide by cos C, the result is

$$\cot A \tan C = \frac{\cot a}{\cos C} \sin b - \cos b;$$

and if we substitute $\cot x$ for $\frac{\cot a}{\cos C}$, (which is equivalent to substituting $\tan x$ for $\cos C \tan a$,) then

$$\cot A \tan C = \cot x \sin b - \cos b$$
$$= \frac{\sin (b - x)}{\sin x};$$

whence $$\cot A=\frac{\sin(b-x)}{\tan C\sin x},$$

or $$\tan A=\frac{\tan C\sin x}{\sin(b-x)}.$$

We have then for calculations in this case the formulæ

$$\begin{cases}\tan x=\cos C\tan a\\ \tan A=\dfrac{\tan C\sin x}{\sin(b-x)}.\end{cases}$$

Thirdly. If the two angles A and B are required; they may be found by means of the formulæ (1), (2), in Art. 37; namely,

$$\tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C$$

$$\tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C;$$

which give half the sum and half the difference of the angles; whence the angles themselves are at once obtained.

Fourthly. If the side c, and A one of the unknown angles be sought; we may find c as above, by the formulæ

$$\tan x=\tan a\cos C$$

$$\cos c=\frac{\cos a\cos(b-x)}{\cos x},$$

and then find A by the additional equation,

$$\cos A=\cot c\tan(b-x),$$

which may be deduced from the second equation (not numbered) in Art. 31, combined with the two others here given.

Fifthly. If the angles A, B, and the side c are all required, the angles may be found by Napier's Analogies or equations as above, and the side c in a corresponding way, by the formula (4) of Art. 37, slightly altered; thus

$$\tan \tfrac{1}{2}c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b);$$

or the side c may be found from the equation

$$\sin c = \sin a \frac{\sin C}{\sin A}.$$

Case 3.

41. *Given two angles* A, B, *and* c *the intermediate side, to find one or more of the remaining parts,* a, b, C.

Here the given quantities are subject only to the limitation that each be less than 180°.

First. If only the angle C be sought, it may be found thus:

(1) From the equation (3) in Art. 31,

$$\cos C = \cos A\,(-\cos B + \sin B \tan A \cos c),$$

and if $\cot x$ be put for $\tan A \cos c$,

$$\cos C = \cos A\,(-\cos B + \sin B \cot x)$$

$$= \cos A \frac{\sin (B-x)}{\sin x};$$

whence we have for computation these formulæ,

$$\begin{cases} \cot x = \tan A \cos c \\ \cos C = \dfrac{\cos A \sin (B-x)}{\sin x}. \end{cases}$$

(2) Also from the equation (3) in Art. 31, we may deduce the equation

$$\cos C = \cos A \cos B\,(\tan A \tan B \cos c - 1),$$

which may be replaced by the two following, that indicate a second method of finding the angle C;

$$\begin{cases} x = \tan A \tan B \cos c \\ \cos C = \cos A \cos B\,(x-1). \end{cases}$$

Secondly. If one of the unknown sides a only is sought, we may deduce it from equation (5) of Art. 30, thus:

$$\cot a \sin c = \cos c \cos B + \sin B \cot A,$$

$$\cot a = \cot c \left(\cos B + \sin B \frac{\cot A}{\cos c}\right),$$

and if $\cot x$ be put for $\dfrac{\cot A}{\cos c}$,

$$\cot a = \cot c\,(\cos B + \sin B \cot x)$$

$$= \cot c \frac{\sin (B+x)}{\sin x};$$

whence obviously we have for use in computation these formulæ,

$$\begin{cases} \tan x = \cos c \tan A \\ \tan a = \tan c \dfrac{\sin x}{\sin (B+x)}. \end{cases}$$

Thirdly. If the two sides a and b are sought, we may compute them by the formulæ (3), (4), of Art. 37; namely,

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c;$$

these give half the sum and half the difference of the sides, from which the sides themselves are directly obtained.

Fourthly. If the angle C, and *a* one of the unknown sides be required; we may find C as above by the formulæ,

$$\cot x = \tan A \cos c$$

$$\cos C = \frac{\cos A \sin (B - x)}{\sin x};$$

and then find *a* by the additional equation,

$$\cos a = \cot C \cot (B - x),$$

or

$$\sec a = \tan C \tan (B - x),$$

which may be readily deduced from preceding formulæ.

Fifthly. If the sides *a*, *b*, and the angle C are all required; the sides may be first found as above by formulæ giving their half sum and half difference, and then the angle C by the equation (2) in Art. 37, thus altered;

$$\tan \tfrac{1}{2}C = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2}(A - B).$$

Or C may be found from the equation

$$\sin C = \sin A \frac{\sin c}{\sin a}.$$

CASE 4.

42. *Given two sides a, b, and an angle* A *opposite the former, to find one or more of the other parts* B, C, *c.*

Here the quantities given must be such that sin *a* be not less than sin A sin *b*; as is obvious from the equation $\sin B \sin a = \sin A \sin b$, since $\sin B$ cannot be greater than unity.

To satisfy this condition, it is sufficient, though it is not necessary, that $\sin a$ be either not less than $\sin A$ or not less than $\sin b$; for neither of these two quantities can be less than their product $\sin A \sin b$, since neither of them is greater than unity.

In the case before us there are sometimes two different triangles to either of which the given quantities may belong; and it is important to determine under what conditions this will happen.

From the equation

$$\sin B = \frac{\sin A \sin b}{\sin a}.$$

it is obvious that if B admits of two different values, they however have the same sine, and must therefore be supplements of each other.

To ascertain when B can have two such values, we will make use of the equation

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

and assume the following

$$\tan x = \cos A \tan b,$$

x being greater or less than 90°, according as the values of A and b are on opposite sides of 90°, or on the same side.

Then
$$\frac{\cos(c \backsim x)}{\cos x} = \frac{\cos a}{\cos b}.$$

The first member of this equation may be made to have any value between the limits -1 and $\sec x$, by attributing to c a proper value greater than x; and it may be made to have any value between the limits 1 and $\sec x$, by giving c some value less than x. And the value in the one case may accordingly be made the same as in the other, unless it fall between 1 and -1; when this will evidently be impossible. Now $\frac{\cos (c \sim x)}{\cos x}$ equals $\frac{\cos a}{\cos b}$, and this falls between the limits 1 and -1, or does not, according as the value of a is nearer to 90° than that of b, or is not.

Hence if a be nearer than b to 90°, c can have but one value; otherwise it will admit of two.

We may now give the formulæ for calculation, that belong to this case.

First. If the angle B opposite the side b is sought, it may be found from the equation

$$\sin B = \sin a \sin b \div \sin a.$$

Secondly. If the side c be required, it may be found by the formulæ used above, namely,

$$\tan x = \cos A \tan b$$

$$\cos (c \sim x) = \cos x \cos a \div \cos b;$$ for c is readily obtained when x and $(c \sim x)$ are known.

Thirdly. If the angle C be sought, it may be found from equation (2) of Art. 30, namely,

$$\cot a \sin b = \cos b \cos C + \sin C \cot A.$$

This may be put under the form

$$\cot a \tan b = \cos C + \sin C \div \cos b \tan A;$$

and if $\tan x$ be substituted for $\cos b \tan A$, then

$$\cot a \tan b = \sin (C + x) \div \sin x.$$

Hence we have these two equations to use in calculation,

$$\tan x = \cos b \tan A, \quad \sin (C + x) = \sin x \cot a \tan b.$$

Fourthly. If the angle B and one of the other parts C and c be required;

(1) We may find B as above; and then find the other required part by one of these formulæ, taken from Art. 37.

$$\tan \tfrac{1}{2}C = \frac{\sin \tfrac{1}{2}(a-b)}{\sin \tfrac{1}{2}(a+b)} \cot \tfrac{1}{2}(A-B)$$

$$\tan \tfrac{1}{2}c = \frac{\sin \tfrac{1}{2}(A+B)}{\sin \tfrac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b).$$

(2) We may find B as before; and then find C by the formulæ

$$\tan x = \cos b \tan A, \qquad \tan (C+x) = -\cos a \tan B,$$

or find c by these formulæ,

$$\tan x = \cos A \tan b, \qquad \tan (c-x) = \cos B \tan a.$$

Fifthly. If the parts required be C and c; we may find one of them by formulæ given above, and then find the other by the equation

$$\sin C \div \sin c = \sin A \div \sin a.$$

Lastly. If all the parts B, C, and c are required; we may first find B by the formula

$$\sin B = \sin A \sin b \div \sin a,$$

and then find the other parts by the formulæ

$$\tan \tfrac{1}{2}C = \frac{\sin \tfrac{1}{2}(a-b)}{\sin \tfrac{1}{2}(a+b)} \cot \tfrac{1}{2}(A-B)$$

$$\tan \tfrac{1}{2}c = \frac{\sin \tfrac{1}{2}(A+B)}{\sin \tfrac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b),$$

or find one of the parts by one of these formulæ, and find the other by the equation

$$\sin C \div \sin c = \sin A \div \sin a.$$

The formulæ for the remaining cases, namely the fifth and sixth, being obtained in nearly the same way with those of the fourth and first respectively, will be given now without investigation.

CASE 5.

43. *Given two angles* A, B, *and an opposite side* a, *to find one or more of the other parts*, b, c, C.

In this case, the parts given must be such that sin A be not less than sin B sin a. And here as in the last case, each of the required parts may admit of two values. *This will happen only when* B *is nearer than* A *to* 90°.

First. To find b use the formula

$$\sin b = \sin a \sin B \div \sin A.$$

Secondly. To find C use these formulæ

$$\tan x = \cos a \tan B, \quad \cos(C+x) = -\cos x \cos A \sec B.$$

Thirdly. To find c use the formulæ

$$\tan x = \cos B \tan a, \quad \sin(c-x) = \sin x \tan B \cot A.$$

Fourthly. To find b and either c or C:

(1) Use the formula $\sin b = \sin a \sin B \div \sin A$, and one of the following

$$\tan \tfrac{1}{2}c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b)$$

$$\tan \tfrac{1}{2}C = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2}(A-B).$$

(2) Find b as before; and then find c by the formulæ

$$\tan x = \cos B \tan a, \quad \tan(c-x) = \cos A \tan b;$$

or find C by the formulæ

$$\tan x = \cos a \tan B, \quad \tan(C+x) = -\cos b \tan A.$$

Fifthly. When c and C are required; find one of them by the method given first, and the other by the equation

$$\sin c \div \sin C = \sin a \div \sin A.$$

Lastly. When b, c, C, are all required; find b by the equation

$$\sin b = \sin a \sin B \div \sin A,$$

and find the other parts by the formulæ

$$\tan \tfrac{1}{2}c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b),$$

$$\tan \tfrac{1}{2}C = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2}(A-B);$$

or find one part by one of these formulæ, and the other by the equation, $\sin c \div \sin C = \sin a \div \sin A$.

Case 6.

44. *Given the angles* A, B, C, *of a spherical triangle; to find one or more of the sides* a, b, c.

The conditions to be observed respecting the quantities here given are these:

(1) $A+B+C > 180°$, (2) $A+B < 180° + C$,

supposing C to be the least angle.

First. For finding only one side a, either of the following means may be used:

(1) $$\cos \tfrac{1}{2} a = \sqrt{\frac{\sin (B-H') \sin (C-H')}{\sin B \sin C}} \text{ (Art. 33)},$$

where $H' = \tfrac{1}{2}(A+B+C) - 90°$.

(2) $$\sin \tfrac{1}{2} a = \sqrt{\frac{\sin H' \sin (A-H')}{\sin B \sin C}}$$

(3) $$\tan \tfrac{1}{2} a = \sqrt{\frac{\sin H' \sin (A-H')}{\sin (B-H') \sin (C-H')}}$$

(4) $$\sin a = \frac{2\sqrt{\sin H' \sin (A-H') \sin (B-H') \sin (C-H')}}{\sin B \sin C}.$$

(5) $$\begin{cases} x = \cos A \sec B \sec C \\ \cos a = \cot B \cot C\,(x+1). \end{cases}$$

Secondly. If two sides a and b are sought;

(1) We may find a as before, and b in like manner, with the proper change of letters.

(2) We may find a as before, and b by the equation

$$\sin b = \sin a \sin B \div \sin A.$$

(3) The sides a and b may be found by these formulæ,

$$\begin{cases} \tan x = \tan \tfrac{1}{2}C \tan \tfrac{1}{2}(A+B) \tan \tfrac{1}{2}(A-B) \\ \cos b = \cot A \cot (\tfrac{1}{2}C - x), \text{ or } \sec b = \tan A \tan (\tfrac{1}{2}C - x) \\ \cos a = \cot B \cot (\tfrac{1}{2}C + x), \text{ or } \sec a = \tan B \tan (\tfrac{1}{2}C + x). \end{cases}$$

Thirdly. If all the sides are sought; then after computing a and b by either of the preceding methods; c may be found by either of the formulæ at the beginning of this article, or by the equation

$$\sin c = \sin a \sin C \div \sin A.$$

Another method of finding the three sides is to compute a by one of the first five formulæ given above, and then obtain b and c from the equations

$$\tan \tfrac{1}{2}(b+c) = \frac{\cos \tfrac{1}{2}(B-C)}{\cos \tfrac{1}{2}(B+C)} \tan \tfrac{1}{2}a,$$

$$\tan \tfrac{1}{2}(b-c) = \frac{\sin \tfrac{1}{2}(B-C)}{\sin \tfrac{1}{2}(B+C)} \tan \tfrac{1}{2}a.$$

This method is most convenient when a is computed by means of the equation

$$\tan \tfrac{1}{2}a = \sqrt{\frac{\sin H' \sin (A-H')}{\sin (B-H') \sin (C-H')}}.$$

A similar method might have been given in the first case, for finding the angles of a triangle from its sides.